Why can't we do it this way?

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Students bring to the classroom shards of information and algorithms learned (or seemingly learned) and they question why they must master new approaches. Yet they seem quite aware that their learning seems insufficient to provide consistent mastery of some topics. Masked behind these inquiries, I find, is a desire to express and explore a personal theory they have evolved (quietly). Probing their silence is a provocative experience. Examples of such positing are presented and discussed. But I also propose that we, as math educators, need to examine why we don't offer alternatives to some of the classical textbook conceptions. Their questions need to become ours. Fundamentally, why don't we use mathematics to explore some alternatives to the classical approaches to learning mathematics?

Introduction

Math requires two things: imagination and rigorous logic ... Kant
Logic will get you from A to B. Imagination will take you everywhere ... Einstein
Most math thinking begins with vague visual images, later formalized in symbols ... Hadamard
The mathematician's feeling of triumph, as he forces the world to obey the laws his imagination has freely created, feeds on its own success ... Rota

Mostly our instructional competence involves teaching students to think with some rigor, some logic and some efficiency of problem solving. We prefer to work in this direction because, historically, math has provided paths for us to follow and these paths are populated with algorithms, rules, and methods for solving problems with the most efficient strategies.

For example, we have the quadratic formula which is a most generalized way to solve quadratic equations. One student, however, also insisted that it could be used to factor trinomials. She was applying Algebra to Algebra and this is the essence of what is to be explored today. Kant, Einstein, and Hadamard (see above) propose that the logic and symbols of abstraction we use to convey math to students have the equally essential component of imagination. But is this addressed when the opportunity arises?

In a classroom of 20 or so people, including the instructor, there is a range of problem solving strategies, many of which have not been voiced. We tend, when we teach, to travel the roads paved for us by generations of mathematicians and curriculum and textbook developers. Not only do we have the quadratic formula but we have the Pythagorean relationship, we have standardized notation, we have the tried and true 'word' problem strategies.

But there are in a classroom students with "…vague visual images…" that can be shaped into interesting, valid and refreshing approaches. These images typically come in the form of "Isn't there an easier way to do this?", which isn't expressing imagination rather it is hiding it. Understandably, these conceptions may not fit the ideal of the most efficient algorithm, but the creativity and imagination involved are, in my view, essential to explore.

Allowing this to happen in the classroom violates, in some cases, the proposed curriculum and time table for 'getting through the material'. But exploring with students some of the imaginative strategies they have been harboring and incubating gives a first hand demonstration of the power of applying Algebra to Algebra.

In *Letters to a young mathematician* (Stewart, 2006, p. 55) Stewart notes: "The traditions of scientific publication (and of textbook writing) require that the 'aha' moment be concealed, and the discovery presented as a purely rational deduction from known premises." And haven't we all had these 'wow' 'aha' moments? Further, haven't we all found at least one colleague with whom we shared this 'aha' moment? Why not have them in the classroom, as the instructor or the student? I do contend based on my experience, and I do suppose based on your experience, that they are more frequent than expected.

Where these insights and intuitions come from is not the focus of this paper. The focus is that they do occur and they need to be included in the curriculum. Of course we can't include that which we aren't sure is going to happen, but if we build in space and time to play with it when it occurs, be assured it will occur.

These moments are aptly described in *The Number Sense* (Dehaene, 1997, p. 119), where children's quantitative capability is described:

> In the first six or seven years of life, a profusion of calculation algorithms see the light. Young children reinvent arithmetic. Spontaneously or by imitating their peers, they imagine new strategies for calculation. They also learn to select the best strategy for each problem. The majority of their strategies are base on counting, with or without words, with or without fingers. Children, often discover them by themselves, even before they are taught to calculate.

Upon reading this, I had an 'aha' moment. I imagined that this capability extended into adulthood. I imagined that it is never abandoned, particularly in the math classroom where it is likely to compete with the traditional algorithms for solving an equation, or factoring a trinomial, or employing a formula. I imagined that the silence instructors sometimes are met with when presenting a new concept is nothing more than students saying, quietly to themselves – and sometimes out loud in frustrating tones – "Isn't there an easier way to do this? Why can't we do it this way?"

And notice the similarity between Dehaene's description and: "Once a theoretical idea has been acquired, one does well to hold fast to it until it leads to an untenable conclusion." (Einstein, 1982, p. 343). It seems that some students may not find a 'C' grade untenable and therefore see no need to give up their theory of how math works! But their theory isn't necessarily bad; rather it's likely to be incomplete and looking at their work, correct and incorrect, doesn't necessarily point us to that theory. It is an assumption, but I contend a fair one, that students may not be aware of holding a theory, so we have to listen and observe carefully to see what their imagination has crafted. Then once we hear what is really being said, we can then explore their idea. Their way has to be heard, then voiced and then explored.

Based on this premise, look at some of the outcomes. The examples presented have been prompted by student questions. Some questions led to an interesting conclusion and some have no conclusion to date.

Let me have your participation. This is not a test nor an experiment but a stepping stone to exploring the idea that strategies other than the classical, efficient modes can be fun and equally as powerful as a teaching moment. As Kant noted, both imagination and logic are to be employed.

So, take a few moments to solve the following problem, presented in *Introduction to Algebra* (W.C. Colburn, 1839, p. 40): A person in play lost 1/4 of his money, and then won 3 shillings; after which he lost 1/3 of what he then had; and this done, found that he had but 12 shillings remaining; what had he at first?

I suspect there are variations on the basic 'identify-the-variable-write-and-solve-the-equation' strategy. One way is to write the equation as one reads:

$$x - \frac{1}{4}x + 3 - \frac{1}{3}\left(x - \frac{1}{4}x + 3\right) = 12$$

A student presented the following as the answer:

6 6 <u>6</u> 18 − 3 5 5 5 <u>5</u> 20

There were other various little scratching and cross-outs on the paper, but the third 6 and the fourth 5 were underlined. The 20 was circled as the answer, and it is correct. Recalling Hadamard's "… vague visual images, later formalized in symbols", this students' solution reflects this concept with some precision. This student was also very proficient with classical algebraic solutions, but as she noted, "I like to play around with ideas". Her question was "Can I solve this problem starting with the result?" Her solution strategy is described in Appendix A.

The dilemma for the instructor is that this was an exercise in solving application problems using Algebra and this student did not use Algebra but a Hadamard model. Does the student earn credit for the correct solution, although procedurally there was the pedagogical expectation of an Algebraic solution strategy?

Let me ask you to again participate in a demonstration. Again, this is neither a test nor an experiment and if you choose, you can work together and share the effort to do the following:

Factor the trinomial $6x^2 + 5x - 4$

The classical solution, as prescribed by most texts, is to first look for a greatest common factor (GCF). In this case, there is none. The next steps may vary, but generally factoring is addressed by (a) trial and error, playing with the combination of factors for 6 and −4, or (b) use of the 'ac' method, also called the grouping method.

An alternative method is titled the blended method and Appendix B contains a derivation. The solution to the problem is $(2x - 1)(3x + 4)$ and the blended method looks like this:

$$6x^2 + 5x - 4 = (6x + 8)(6x - 3) = (3x + 4)(2x - 1)$$

The point here is not to discuss details of this blended method but to note that the second statement is a departure from expectation, in that the first term is the same in both binomials. In the third statement, the first binomial has a factor of 2 removed, the second binomial has a factor of 3 removed, but these factors are not included in the solution. They simply disappear. Students and I worked through this method. There were several false starts but the key was to allow students to participate in its development. They posed alternatives and offered "can we do this?', "why won't that work?', "what if we tried …", "are you allowed to do that?" and other imaginative attempts to apply Algebra to Algebra.

Why can't we do it this way? We can and I and several of my colleagues have and, importantly, students have responded positively to it. This method is really applying Algebra to Algebra in the sense that the essence of Algebra isn't to solve equations, but, in my view, rather to identify, connect and then summarize quantifiable relationships in the most general of terms.

This blended method simplifies factoring trinomials and 'special' cases, although not all conceivable factoring problems are covered. But consider the following: One textbook consumes about 40 pages discussing and demonstrating a hierarchy of factoring strategies. Other texts average about 30 or so pages. The blended method, if written textbook style, perhaps would fill a page. Respectfully, although the blended method omits a discussion of some of the special cases, coverage of these special cases would not require an additional 30 to 40 pages of text.

The issue is not to rewrite the texts but to rethink how we might use Algebra to do Algebra; we seemingly don't use our imagination nor facilitate students using theirs. If Algebra is a conception for providing an efficient strategy for problem solving, the contention is that both imagination and logic need to be employed. There is no forgone conclusion that this leads to a 'better' method, but it does provide students with a sense that their thinking is productive and effective. If we, as mathematicians and math educators are exhilarated by seeing and then formalizing a relationship, why not include students in this process? As Rota noted: "The mathematician's feeling of triumph … feeds on its own success". But that feeling should be available not only to mathematicians, but also to students in mathematics classrooms.

Here's another example of using Algebra to do Algebra and seeing a student's question as imagination, not aggravation. This type of problem (slightly modified) may not be universal, but it is consistently offered in U.S. texts. Three examples are provided for the purpose of discussion. Appendix C contains a derivation of this strategy.

1. Carl does a job in 4 hours. Paul does it in 2 hours. Together, how long will it take?

2. Bill does a job in 3 hours. John does it in 5 hours. Together, how long will it take?

3. Mary does a job in 2 hours. Jane does it in 3 hours. Together, how long will it take?

Problem (C) can be solved using one of the classical strategies, and would look like this:

$$\frac{x}{\text{Mary's time}} + \frac{x}{\text{Jane's time}} = 1\,(\text{job done})$$

Set up as: $\frac{x}{2} + \frac{x}{3} = 1$ and solved, $x = 1\frac{1}{5}$ hours.

Problems (A), (B) and (C) would employ exactly the same solution methodology, therefore as a student proposed - Why can’t we do it this way?

Since the algebraic approach – establish the unknowns, identify the relationships, write and solve an equation – is the same in every case, rather than using the numbers in each case, use a general solution, namely:

Set up as: $\frac{x}{a} + \frac{x}{b} = 1$ and solved $x = \frac{ab}{a+b}$

At this point, doing it this way seems more algebraically efficient, but we can’t lose sight of our being in the classroom. The purpose of deriving these algorithms is not to provide students with the ‘easiest’ solution strategy. The purpose is to have students work through these kinds of derivations in the classroom and realize that employing their basic understanding of Algebra and solving equations enables them to extend their capability for handling quantitative solutions. Further, having them realize that their questions can lead to such conclusions is a very satisfactory ‘aha’ moment.

Students with a mastery of solving equations can be urged to direct their energies in a creative way – using their imagination – to identify and where possible, derive a general algebraic solution to a class of problems, rather than simply employing an algorithm provided by historical mathematical precedent. I understand the importance of extending the culture of mathematics from past to present. But, enabling students to employ the concept of Algebra beyond the algorithms of the text - as valid and as important as they are – and to derive their own algorithms seems a more acceptable goal than knowing that students have learned, by rote or by practice, only the prescribed curriculum.

And to the point: students struggled to derive the above ‘shared time’ relationship, but literally cheered upon finding it and then used it to readily solve the remaining problems in the exercise. When presented with a problem which gave the shared time and the time of one of the people and asked to find the time of the second person, they realized that the algorithm worked. One student noted that it was easier to see where the shared time and the individual times fit in their rule rather than the way proposed by the text. It is to be noted that the student used the phrase "their rule", demonstrating ownership of the procedure that had come about.

I view this process as critical to students connecting to the historical threads of mathematics development, to experiencing the 'aha' moment, the 'wow' of taking a mass of seemingly disconnected material or material fixed in form and function, and generating a well-constructed formulation. I contend that it's not only what they know but also how they arrive at that knowing that will move them more comfortably along in learning and applying math.

This is also a potentially worrisome experience for an instructor. Students do present solutions which seemingly have no logic, only imagination. The need is then to work through the scribbling and the "…vague visual images…" and help the student formalize what has occurred. This is worrisome because, as has happened, the student can't verbalize it. A brief story: 'Word' problems were demonstrated as being solved by using either the addition method or the substitution method with two equations with two unknowns. One student solved all the problems correctly without ever writing anything. He stared at the problem for a time and then announced the answer. He was asked to verbalize what he was doing while he solved the problem. He couldn't do it. More critically, while attempting to state the process he used, he couldn't get a problem correct. I was again presented with the dilemma of a student not demonstrating the required mastery of the expected procedure yet able to solve problems correctly.

One more example of how an aggregation of student's imaginations led the class, including the instructor, to apply Algebra to Algebra. A type of problem which frequents texts in the U.S. is the 'mixture' problem. Mixing solutions of different concentrations to get a third with a desired concentration; mixing different kinds of candy or nuts to get a mixture to sell at a certain price, or getting a return on investing in two accounts at different interest rates (and other applications). I again ask for your participation. Take a few moments to work on this problem:

> Gus has on hand a 5% alcohol solution and a 20% alcohol solution. He needs 30 liters of a 10% alcohol solution. How many liters of each solution should he mix together to obtain the 30 liters?

The classical solution is to write the equation $.05x + .20(30 – x) = .10 (30)$. Solving the equation, the outcome is 20 liters at 5% and 10 liters at 20%.

As this type of problem was discussed, one student asked "can <u>15+x</u> and <u>15-x</u> be used, since they add to the total of 30 and this seems easier?" I wasn't certain if it would work nor why it seemed easier, but we explored the idea. The student presented the problem on the board as $5(15 + x) + 20(15 – x) = 10 (30)$. Not only did he re-craft the unknowns, but he used whole numbers, not decimals. When asked why, he simply stated that we would be getting rid of the decimals anyway. I noted that 'getting rid of' is not a mathematical operation, but clearly it works. The answer to this equation is -5, and some students believed this solution to be awkward for two reasons. First because $x = -5$, and a negative quantity doesn't make sense and second, because finding x doesn't finish the problem. Recall that the unknowns in the equation are <u>15+x</u> and <u>15–x</u>, so another step is required to come to the correct answer of 20 liters at

5% and 10 liters at 20%. Then the question came: "Isn't there another way to do this?"

Another student announced "Sure, since we cut 30 in half, just double the unknown". This was odd, but very imaginative and if the equation is set up with the unknowns as <u>15+2x</u> and <u>15–2x</u>, it works, but again, there is that additional step to get the final answer.
This prompted another student to propose that rather than messing with things like this, we should do Algebra with Algebra and set up this type of problem as a general statement.. The students were assigned the work of solving the general statement but were uncertain of the meaning of the answer. After 'messing with it' in class, the resolution is as noted in Appendix D, which contains a derivation of an alternative solution strategy for this type of problem.

It looks like this:

$$Q_1 = T \bullet \frac{|P_3 - P_2|}{|P_2 - P_1|} = 30 \bullet \frac{|14-18|}{|18-12|} = 30 \bullet \frac{|-4|}{|6|} = 30 \bullet \left(\frac{4}{6}\right) = 20$$

Where Q1 = the quantity to find for P1, T = total amount given, P3 = percent associated with the given amount, P2 = one of the percents of a quantity to find, and P1 = one of the percents of a quantity to find.

Who would have imagined such a relationship?

The conclusion of this paper is in the initial four citations regarding imagination. We, as educators, don't have to sit up late at night creating imaginative algorithms for students. Rather, what is being proposed is to listen and observe carefully for those moments when a students' imagination trickles through the influence of mathematical precedent. Not all trickles can become torrents, but every class offers the promise of personal transcendence and I hope to be there to share in the experience.

References

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Dehaene, S. (1999). *The Number Sense: How the Mind Creates Mathematics.* New York: Oxford University Press.

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Appendix A. Hannah solves an application problem

A problem presented in class from Introduction to Algebra (W.C. Colburn, 1839, p. 40) was:

A person in play lost 1/4 of his money, and then won 3 shillings; after which he lost 1/3 of what he then had; and this done, found that he had but 12 shillings remaining; what had he at first?

The expectation was that students would apply a problem-solving strategy we had studied, namely, identify the unknown, identify the relationships and write and solve an equation. Hannah, however, presented the following:

$6 \quad 6 \quad \underline{6} \qquad\qquad 18-3 \qquad\qquad 5 \quad 5 \quad 5 \quad \underline{5} \qquad\qquad \boxed{20}$

The underlines and box are the students' and 20 is the correct answer. When asked how she came to this answer and the meaning of the string of numbers, her explanation was (paraphrased):

I worked backwards. I started with the answer 12, and 12 was 2/3 of something since 1/3 was taken away. If it was 2/3 then 2 equal parts of the 3 equal parts of 12 is 6 each, so the three 6's gives 18. He gained 3 shillings, so I had to take three shillings away, so I got 18-3 or 15. And then I did the same thing with 15 that I did with 12. Since 15 was 3/4 of something since 1/4 was taken away, 15 is 3 equal parts of the 4 things, or 3 equal parts of 5 each. Since there are 4 equal parts to start, I added one more 5, and the answer is 20.

This student, by the way, is very proficient in solving equations and very good at applying traditional strategies to problem solving. She just chose to play with this problem this way.

Appendix B: Factoring - A Blended Method

When factoring is introduced as a topic, it typically goes through several iterations. These iterations identify conditions that lead the student to apply different factoring strategies. Most notably this happens when dealing with factoring trinomials because factoring trinomials leads to solving quadratic equations by factoring.

The different strategies emerge as a function of identifying certain characteristics of the trinomial. To highlight this, the following from a textbook are headings for the sections in the chapter on factoring polynomials: (1) the greatest common factor and factoring by grouping, (2) Factoring trinomials whose leading coefficient is one, (3) factoring trinomials whose leading coefficient is not one, (4) the 'ac' method, (5) factoring special forms, and (6) a general factoring strategy. Students not only have to identify the conditions which exist in the problem (and they may be overlapping!), but they also have to know which conditions are most amenable to which form of factoring.

One constant consideration is to always start by looking for and factoring out the greatest common factor (GCF). Then the student is to identify the trinomial as a candidate for factoring by (1) trial and error, (2) grouping, (3) a special form, (4) or a mix of strategies.

There is, however, one basic way to factor which underpins all these various strategies. In essence, there is one strategy for factoring trinomials, not the various ways noted above.

Before showing examples please note that this method is crafted from elements of all the above strategies. This method for factoring trinomials also applies to some special cases, although some conditions need further exploration (for example, the difference of two squares and perfect square trinomials, beyond the 2nd degree).

The blended method: Using a generic example, start with a trinomial of the form $ax^2 + bx + c$. After factoring out the GCF, the student is to find 'ac'. The product 'ac' is found by multiplying the coefficient of the x^2 term by the constant, thus 'ac'. This is actually a procedure within the grouping method and the grouping method is sometimes referred to as the 'ac method'. In some cases, if the values 'a' and 'c' are larger numbers, pairs of factors of both can be noted, rather than finding the product. An example is provided below. The next thing to find are the two factors of the product 'ac' which when added together give 'b' – the coefficient of the 'x' term in the trinomial.

Call these two factors 'm' and 'n'. Write 2 sets of parentheses, indicating multiplication between them, thus (   )(   ). This is the classical factoring of a trinomial into 2 binomials. If a trinomial is prime (can't be factored), it will be identified at this point by not finding 'm' and 'n' which add to 'b'.

Place the term 'ax' (not ax^2) as the first term in each of the parentheses. In this example, we would have (ax   )(ax   ). Next place 'm', including its sign, as the second term in the first parenthesis and place 'n', including its sign, as the second term in the second parenthesis. It is to be noted that the 'm' and 'n' terms can be placed in either the first or second parentheses. At this point, they are interchangeable. The two binomials are $(ax + m)(ax + n)$.

This is unlike the classical methods of factoring. The 'a' term would be accounted for in the 'ac' term. Either by using the grouping method or by trial and error, various sets of factors of both the 'a' coefficient and the 'c' constant term would be explored to identify the coefficient of the first terms and the constant in each parenthesis. In the blended method, once the factors of 'ac' that add to 'b' are known, the two binomials are formed.

In $(ax + m)(ax + n)$ either 'm' or 'n' or both may be negative. If there is a common factor in the first binomial, it is to be divided out, not factored out. For example, if the binomial were $3x + 12$, the common factor of 3 would be divided out, leaving $x + 4$. Call this factor c_1. This common factor of 3 is not part of the solution. Repeat this process for the second binomial. If there is a common factor, it is to be divided out. And again, this common factor is to be noted but is not part of the solution. Call it c_2. It may be that neither binomial has a common factor, but both binomials must be examined. Once the common factor of each binomial (if one exists in each) is divided out the result is the solution, given the extraction of a GCF. The role of c_1 and c_2 will be noted in the following summary.

Table 1 The Blended Method

Given a trinomial of the form $ax^2 + bx + c$: If there is a greatest common factor, factor it out. It is part of the solution. Multiply ‘a’ and ‘c’, producing the product ‘ac’. Or, start with pairs of factors for ‘a’ and ‘c’. Find the two factors of ‘ac’ that added together give ‘b’. Name them ‘m’ and ‘n’. Write the two binomials (ax + m)(ax + n), noting that ‘m’ and ‘n’ might also be negative. Extract any common factor in the first binomial. Call it c_1. Do the same for the second binomial and call it c_2. Neither c_1 nor c_2 are factors in the solution. The solution is the two binomials, after step 5 is done, including any GCF from step 1. To check for the presence of a GCF (step 1 not done nor complete) multiply c_1 and c_2 from step 5; divide this product by ‘a’ in step 4. If it equals 1, factoring is complete. If it is a number other than 1, that number must be included as a factor in the solution. It is either the GCF or a common factor missed in step 1.

Please note this critical element about the factors c_1 and c_2 that are divided-out in step 5 above. Since the first and second binomials in step 4 both have the ‘a’ value, the factors that are divided out will multiply together to get this ‘a’ value, if the GCF was extracted completely in the first step. This is so because an extra ‘a’ was introduced by including it as the coefficient of the ‘x’ term in both binomials. Please note that this check on the extraction of the GCF (step 7) is not essential to the blended method if extracting the GCF is always done, but it can be seen as a way to verify that the GCF was extracted as a first step.

Look at the following examples. They are demonstrated step by step, as noted in Table 1.

Example 1: Factor $8x^2 + 12x - 8$.

Step 1: Extract the GCF of 4, giving $4(2x^2 + 3x - 2)$.

Step 2: ‘ac’ = $(2)(-2) = -4$

Step 3: The factors of -4 that add to 3 are 4 and -1.

Step 4: The two binomials are $(2x + 4)(2x - 1)$

Step 5: $2 \cdot (x + 2)(2x - 1)$. This factor of 2 is NOT part of the solution. At this point, the solution is $(x + 2)(2x - 1)$.

Step 6: the factors are $4(x + 1)(x + 2)$. The factor 4 from step 1, and the binomials

$(x + 2)$ and $(2x - 1)$ from step 5. (Note that the divided-out factor in step 5 is not part of the solution)

Step 7: The product of the numerical factors in step 5 is 2. Dividing this by 2 (the "a" coefficient of "x" in the binomial in step 4), one gets 1, signaling that the GCF was extracted in step 1.

Example 2. Factor $36x^2 + 55x - 14$

Step 1: There is no GCF.

Step 2: 'ac' = 4•9•2•7. Note here that pairs of factors are used for the 'a' and 'c', rather than the product.

Step 3: Playing with these factors (trial and error), it can be seen that 9•7 - 4•2, or 63 – 8 = 55

Step 4: The two binomials are (36x + 63)(36x − 8)

Step 5: Extracting 9 from the first binomial and 4 from the second binomial leaves

(4x + 7)(9x − 2). The 9 and 4 are NOT part of the solution.

Step 6: The factors are (4x + 7)(9x − 2). There is no GCF. To repeat, the factors of '9' and '4' extracted in step 5 are not included in the solution.

Step 7: The product of the numerical factors in step 5 is 36. Dividing this by 36 (the 'a' coefficient of 'x' in the binomial in step 4), one gets 1, signaling that the GCF was extracted in step 1.

Appendix C: Shared Work Time Problems

A common problem presented in Algebra texts states how long it would take each of two people to do a job alone. The student is asked to find how long the job would take if the two people worked together. This type of problem can also be presented with the 'shared' time and the 'working alone' time of one person given, and the student is to find the time for the second person working alone.

For example, Danny takes 4 hours to do a job alone. His brother Mike can do the job alone in 6 hours. If they work together, assuming no gain or loss of efficiency, how long will it take them to do the job?

Classically, $\frac{1}{4} + \frac{1}{6} = \frac{1}{x}$ or $\frac{x}{4} + \frac{x}{6} = 1$

In both cases, the strategy is to set up an equation with fractions and solve for 'x'. Regardless of the actual numbers for the times both individuals work, the formula is the same. Given that, if Algebra is applied to Algebra, we can state the following:

Equation 1: $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$ or Equation 2: $\frac{x}{a} + \frac{x}{b} = 1$

In both cases, solving for x one gets: $\frac{ab}{a+b}$

Call this formula 1. For the problem stated above, the shared time to do the job would be: $x = \frac{a \bullet b}{a+b} \Rightarrow x = \frac{4 \bullet 6}{4+6} \Rightarrow x = 2\text{ hrs., }24\text{ min}$.

If given the time of one individual and the shared time, formula 1 can be used to find the time of the other individual. For example: If Mary and Jane can do a job together in 1 hour and 12 minutes, how long would it take Mary to do it alone, if Jane can do it alone in 2 hours?

1 hour and 12 minutes is $1\frac{1}{5}$ or $\frac{6}{5}$ hours working together. Using formula 1, the problem can be seen as: $\frac{6}{5} = \frac{2a}{2+a}$, the answer being that Mary would take 3 hours working alone.

This is particularly useful, since students seem to struggle with problems having fractions in the numerator or denominator (or both) of a fraction, which would be the case if either of the classical forms in equation 1 or 2 were used. This conception was worked out with students, so that they had ownership of the formula and were aware of the value of applying Algebra to Algebra.

Appendix D: Unmixing Mixture Problems

Mixture problems – solutions of acid / milk / water / butterfat, etc., or nuts/candy/coffee/etc., or two kinds of coins, two types of tickets sold, or interest from two deposits, etc. – can be solved with a generic algorithm, derived from the classical set-up-an-equation-and-solve format. This algorithm comes from applying Algebra to Algebra.

For example, using the problem "how much of a 12% and 18% solution must be mixed to get 30 liters of a 14% solution?", the classical approach would result in the equation:

(Equation 1) $.12x + .18(30-x) = .14(30)$ which gives 20 liters of 12% and 10 liters of 18%.

Applying Algebra (in essence, using the general case), the above equation can be seen as:

(Equation 2) $P1Q1 + P2(T - Q1) = P3T$, where $P1 = .12$, $Q1 = X$, $P2 = .18$, $T = 30$, and $P3 = .14$

Note: any solution mixture problem set up classically would result in a similar equation and therefore a similar set of P's and Q's

Solving Equation 2 for Q1:

$$P1Q1 + P2(T - Q1) = P3T$$

$$P1Q1 - P2Q1 + P2T = P3T$$

$$Q1(P1 - P2) = P3T - P2T$$

$$Q1(P1 - P2) = T(P3 - P2)$$

(Equation 3) $Q1 = T\left(\frac{P_3 - P_2}{P_1 - P_2}\right)$

It is possible that the difference in either the numerator or denominator or both might be negative, therefore the absolute values of both are taken.

Since $|P_3 - P_2| = |P_2 - P_3|$ and also $|P_1 - P_2| = |P_2 - P_1|$, equation 3 can be expressed as

(Equation 4) $$Q_1 = T \bullet \frac{|P_3 - P_2|}{|P_2 - P_1|}$$

The numerator of equation 4 states "take the absolute value of the difference between the percent of the given value and the percent of the value of the second quantity". In this case the given value is 30, with a percent of 14 and the percent of the second quantity Q2 is P2 (or 18).

The denominator of equation 4 states "take the absolute value of the difference between the percent of the second quantity and the percent of the quantity for which you are solving". The percent of the second quantity Q2 is P2 (or 18) and the percent of the quantity Q1 for which you are solving is 12.

These percents – the GIVEN value, the OTHER (P2 in this case) and the quantity TO GET – when always aligned vertically as seen in table 1 can be substituted in equation 4.

Table

Percent of the <u>given</u> quantity	(P_3)	14
Percent of the <u>other</u> quantity	(P_2)	18
Percent of quantity <u>to find</u>	(P_1)	12

Percent notation is not used here, only the values. If equation 4 is now evaluated using these values with T = 30, the result is

$$Q1 = T \bullet \frac{|P_3 - P_2|}{|P_2 - P_1|} = 30 \bullet \frac{|14-18|}{|18-12|} = 30 \bullet \frac{|-4|}{|6|} = 30 \bullet \left(\frac{4}{6}\right) = 20$$

This is the same result found for the 12% solution when solving equation 1.

Had the 18 and 12 percents been switched in Table 1, "the percent to find" would be the quantity for the 18% solution. For this and other problems where two quantities are to be found, it makes no difference which quantity is found first. For example, If the quantity of 18% solution is to be found first, the statement would be:

Percent of the given quantity (Ps) 14

Percent of the other quantity (P2) 12

Percent of quantity to find (P1) 18

$$Q1 = T \bullet \frac{|P_3 - P_2|}{|P_2 - P_1|} = 30 \bullet \frac{|14-12|}{|12-18|} = 30 \bullet \frac{|2|}{|-6|} = 30 \bullet \left(\frac{2}{6}\right) = 10$$

The percent of the given quantity is always the 1st term in the list, the quantity to be found is always the 3rd term and the "other" is always the middle term. The designation Q1 is applied to the quantity to be found.