Adults Learning Mathematics

An International Journal

Chief Editor
Gail FitzSimons

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Objectives

Adults Learning Mathematics – an International Research Forum (see http://www.alm-online.org/) has been established since 1994, with an annual conference and newsletters for members. ALM is an international research forum bringing together researchers and practitioners in adult mathematics/ numeracy teaching and learning in order to promote the learning of mathematics by adults. Since 2000, ALM has been a Company Limited by Guarantee (No.3901346) and a National and Overseas Worldwide Charity under English and Welsh Law (No.1079462). Through the annual ALM conference proceedings and the work of individual members an enormous contribution has been made to making available theoretical and practical research in a field which remains under-researched and under-theorised. Since 2005 ALM also provides an international journal.

Adults Learning Mathematics – an International Journal is an international refereed journal that aims to provide a forum for the online publication of high quality research on the teaching and learning, knowledge and uses of numeracy/mathematics to adults at all levels in a variety of educational sectors. Submitted papers should normally be of interest to an international readership. Contributions focus on issues in the following areas:

- Research and theoretical perspectives in the area of adults learning mathematics/numeracy
- Debate on special issues in the area of adults learning mathematics/numeracy
- Practice: critical analysis of course materials and tasks, policy developments in curriculum and assessment, or data from large-scale tests, nationally and internationally.

The ALM Journal will be published twice a year.
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Information about contributions for the ALM International Journal is available on www.alm-online.org
Books for review, or completed book reviews, should be sent to the Chief Editor.
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// Adults Learning Mathematics - An International Journal

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In this Volume:

Editorial

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Welcome to the first volume of *Adults Learning Mathematics — An International Journal*. As noted on the second cover page, *Adults Learning Mathematics — An International Forum* was established in 1994 and has conducted annual conferences since then in England, Ireland, the Netherlands, the U.S.A., Denmark, Austria, Sweden, and Australia. Not only are all conference proceedings published, but 22 ALM Newsletters have also been issued to March 2005.

In 2000 an edited book, *Perspectives on Adults Learning Mathematics: Research and Practice*, was published by Kluwer Academic Publishers and has attracted much international interest. In addition, many ALM members have published monographs, book chapters, and refereed journal articles, individually or in collaboration with others. Clearly there is a growing interest in issues associated with adult mathematics and/or numeracy education around the world, and the need for a refereed journal to complement these other publications was seen as a priority by Juergen Maasz during his term as Chair of ALM from 2001 to 2004. His enthusiasm for this project was supported by the ALM Trustees and continues under the current Chair, Katherine Safford-Ramus.

We are especially privileged to have also gained the support of an international group of eminent mathematics educators who form the Editorial Board. We are also grateful to the number of ALM members who have volunteered to act as reviewers for articles submitted — in addition to their other demanding academic duties. The contribution of both groups is much appreciated. This journal is to be published twice a year, and the reviewing process for the second issue of Volume 1 is already under way. We look forward to receiving many more interesting and diverse articles addressing issues of relevance to ALM.

Introduction to the contents of the first issue

In *A New View of Mathematics Will Help Mathematics Teachers*, Juergen Maasz from Austria examines from a sociological perspective the history of the discipline of mathematics in northern Europe over the last 200 years. In its inter-relationship with mathematics education, Maasz argues that by seeing mathematics as a social construct teachers will be motivated to make pedagogical decisions in the best interests of their students. The history of mathematics is often taken for granted in times when there are pressures such as covering the mandated curriculum within constraints such as limited time frames, contact hours, and teaching resources — to name but a few. Reflecting on the history of mathematics, and indeed the history of mathematics education, offers teachers a different perspective and the possibility of forming new understandings of how they themselves have been positioned — and continue to be. Recognition of the antecedents offers a first step towards questioning how things might be otherwise, to see some of the tensions and contradictions inherent in any formal mathematics teaching-learning situation, and to move towards their resolution.

In their article based on a Canadian funded research project, *Folding Back and the Growth of Mathematical Understanding in the Workplace*, Lyndon Martin, Lionel LaCroix, and Lynda Fownes explore the attempts of a trainee plumber to solve a pipe-fitting problem involving mathematical concepts of multiplication, fractions, and imperial units. They propose that images held by adults of concepts such as these cannot be assumed to be flexible, deep, or useful in specific workplace contexts. They draw upon the notion of *folding back*, based on a theory developed by Susan Pirie and Tom Kieren, further refined by Lyndon Martin's 1999 doctoral dissertation, to discuss the growth of mathematical understanding in workplace training. They see this theory as offering a way to identify a potential moment of breakdown or disconnection, as well as a means to attempt to overcome these. However, they caution that folding back is not a one-off process but forms part of an evolving development of understanding as images held by the learner become more flexible and general.

In the third article, *Mathematical Autobiography Among College Learners in the United States*, Shandy Hauk analyses the data from a research project involving the assignment of a mathematical autobiographical essay completed by university students. Drawing on a cognitive psychological theoretical foundation, her findings reveal that perceptions of intentional engagement with mathematics were likely to be externally driven, as was the locus of control in mathematics education. Decision making patterns were connected to a range of intimately felt personal factors — an insight likely to resonate with many adult mathematics/numeracy practitioners. Hauk asserts that writing a mathematical autobiography allows students to acknowledge previous stressful and threatened reactions to mathematics, thereby building a reflective awareness of them. Within a supportive learning environment students can then begin to attribute their successes to internal, personal sources, rather than external sources. Hauk concludes with recommendations for further research.

I commend these articles to the reader, and strongly encourage further submissions.

Gail E. FitzSimons
on behalf of the Editorial Team:
Juergen Maasz & Mieke van Groenestijn

A New View of Mathematics Will Help Mathematics Teachers

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Abstract

For many people mathematics is something like a very huge and impressive building. It has a given structure with lots of levels and rooms. For many people this structure and therefore mathematics itself is independent from society, culture and history. It exists and mathematicians try to recover (not: to construct!) new parts of it. From this point of view mathematics is often seen as a lifeless and strange thing and not as a living construct of human beings.

Many mathematics teachers argue that they can't change their way of teaching because they see mathematics from this dominant point of view and think that mathematics will not allow changes. Asking what this means they say that mathematics is something independent from them with a fixed structure. Therefore they have to teach little parts of mathematics (often concentrated on the correct use of algorithms) in a fixed sequence. Changing the sequence or leaving out a part seems to be not allowed. So they are not happy with mathematics but they see no way to change mathematics and therefore no way to change their teaching (perhaps except in some minor important methodical aspects).

I think there is a "way out" if mathematics is seen as a social construct. Is this view correct? A new look at the history of mathematics proves that the history of mathematics in the last 200 years looks like a very good example of applying a sociological theory to make a new interpretation of what happens. In more provocative words: If the sociological theory I apply to make my interpretation of the history of mathematics had existed 200 years ago one could think that the mathematicians tried to prove that the sociological theory is correct by forming the history of mathematics in the way the theory "wants". If teachers try to share this view they will be able to recognize that mathematicians decided what "mathematics" is. I hope this will motivate teachers to make more individual and pedagogical decisions on what and how they teach.

Introduction

In my opinion it is very important for our society to change the image of mathematics, since many studies show that the bad image of mathematics has not changed in past hundred years.
On the one hand many adults remember their experience of learning mathematics at school or university as a bad and stressful situation. They did not understand why they should learn to handle all these difficult algorithms. Many teachers have not answered their important "Why?" - question satisfactorily. Teachers often tell their students: "You will understand this later on, after you have learned mathematics!" Many mathematicians as well as educators and teachers of mathematics have criticised this – for example, Felix Klein about a hundred years ago. But the situation has not changed. If we read Klein today (see Klein, 1926, and his contributions to the "Meraner Reform" analysed by Inhetveen, 1976) we have the feeling that he is talking about today's mathematics lessons.

On the other hand *"too few people recognize, that the high technology so celebrated today is essentially a mathematical technology",* according to Edward E. David, former president of Exxon Research and Development (David, 1984, p. 142). Mathematics is really a powerful force enhancing technological development and changing society and the lives of everyone:

- Mathematics is the basis of the new technologies, since mathematical algorithms are included in all computer software, and computer hardware is materialised mathematical logic.
- Increasingly efficient computers with a great variety of software also play an important part in contributing to the growing influence of mathematics, not only indirectly through other sciences, but also directly through the mathematical models integrated in standard software.
- Mathematical methods and ways of thinking are being used in an ever-increasing number of scientific disciplines (Who is not using statistical methods nowadays?) and, as a result, in more and more spheres of life.
- Mathematics helps to plan the future by constructing models and simulating different development strategies in economics and politics (see Maasz & Schloeglmann, 1988).

The public opinion about mathematics and its real influence on life and society do not match up. Not understanding the sense of learning and the correct ways of applying mathematics reduces the chances of many people. Changing the teaching methods of mathematics would be very helpful in this respect, but it is not enough. Looking at mathematics from a new point of view would open new chances for adults and for the society. What I intend to do with my contribution is to open a new view on mathematics itself. This view might be helpful in changing people's attitude towards mathematics, too.

The Sociological Theory used in this Article

The starting point of my analysis of mathematics from a sociological point of view is the selection of a suitable sociological theory. I think that a version of Niklas Luhmann's theory of social systems would be very useful but this is not the right place to explain this decision by comparing his and other sociological theories. Luhmann (1970, 1984) has developed his theory of social systems over many years and has changed several aspects and definitions. This has lead to an ongoing discussion in sociology and has introduced developments in other disciplines (see, for example, http://en.wikipedia.org/wiki/systems_theory). This paper is structured by my understanding of Luhmann's theory (see Maasz, 1985).

The historical background

If we read Bourbaki we get the impression that there is one and only one mathematics and the history of mathematics is the history of important persons who discovered parts of it. If there is any social influence on mathematics in this view it is concentrated on one aspect: How rich is the society? How much research is funded? This decides how fast new mathematics is discovered. The results of research into mathematics are fixed in Plato's world of ideas before the research starts and it is independent from us. This image of mathematics is one important reason for the authority of mathematics – there is nothing to construct.

I should point out that I am not writing a new history of mathematics. I draw on several historical facts to collect the data I need to illustrate that the history of mathematics has happened as if it follows Luhmann's theory. Or in other words: Luhmann's theory is a good key to understanding what happened over the last two centuries and why. This sociological view gives us much more understanding than a philosophical approach.

In this article I focus on historical facts that happened in the centre of Europe, especially in Preussen (i.e., Prussia). This might appear Eurocentric but I think that many of the most important decisions about mathematics with ongoing consequences really happened then and there.

Links to Adults Learning Mathematics

This article is not intended as course material for a typical group of adults learning mathematics. The argumentation proposed here (for details see Maasz, 1988) is part of an academic debate about mathematics and its relation to society. This debate should be part of studying mathematics at university level. This article is written to suggest to teachers that there is a debate about mathematics and society. This might motivate them to read more about it, or at least to feel more free to change their emotional relation to mathematics, as a precondition to help to change the image of mathematics that adults learning mathematics in their courses have. I do not expect that this article answers all questions arising from reading it. But I hope that we may start to have a discussion about it!

Let us now begin with a look at the situation of mathematics in the 19th century. I want to point out that in this century mathematics was a part of natural sciences, personally and functionally closely connected to physics, astronomy, mechanics and geography. In the 19th century the basic structure of this social system was built up, but the functional differentiation did not happen. Mathematics became a relatively autonomous subsystem of the sciences with its own rules for accepted "truth" in the 20th century.

Looking at the History: The Building up of System Structures to Prepare for the Birth of the Social Subsystem "Mathematics"

In his analysis of science as a social system Luhmann points out the importance of a good solution for the problems of obtaining and processing information. These problems were solved by building up structures of communication including locations to meet informants; to find literature, to exchange information and ideas, to teach and to learn. Such locations were the reformed universities and the academies of 200 years ago.

In addition to this, other structures were useful to exchange information with colleagues in other locations. In earlier centuries scientists wrote letters and travelled to visit colleagues or went to places like monasteries, where documents were stored. It is well known how important letters and visits were for mathematicians like Bernoulli, Fermat, Leibniz, Newton, Euler, and Gauss. In the 19th century new types of communication were institutionalised: Mathematicians founded journals and national organisations like the *Deutsche Mathematiker Vereinigung* (DMV) (i.e., Society of German Mathematicians). Mathematicians started to edit mathematical journals like the *Crelles Journal*, the *Mathematische Annalen* (i.e., Mathematical Yearbook), edited by A. Clebsch and C. Neumann, the *Yearbooks on the Progress in Mathematic*s. In the beginning these journals were filled with papers from different subjects like mathematics, mechanics, physics, geography, engineering, technical sciences (related to mathematics), and others. For many years the journals were not specialized in mathematics or parts of mathematics like they are today.

This is important because a Robinson Crusoe cannot build a social system. Systems are built by a lot of social interactions that constitute a continuous social activity. Inner structures of a social system are a better way to make sure that such activities are not lost or wasted. The development and the quality of science as a social system very much depends on the quality and density of information exchange. In earlier centuries the main problem of today – which is called "information overload" caused by the electronic communication infrastructure and the use we make of it – was not known. Mathematicians had great difficulty getting information about the results of their colleagues' work from anywhere else, even from a neighbouring town. Coming together at locations, attending conferences, and reading journals were important steps towards making communication more effective.

I will now summarize some aspects of this development. The first aspect seems simple: A social system needs persons who are qualified to take part in the specific interaction. Mathematics has always needed and still needs an organised process of teaching and learning mathematics. Or, in other words, there is no social system "mathematics" without mathematicians.

About 200 years ago a typical university consisted of faculties like theology, medicine, philosophy and law. Natural sciences and, as a component, mathematics were located in the philosophy faculty. Only very few people were able to study at that time. Students needed wealthy parents or another sponsor to pay for their studies. Only very few students attended lectures about mathematics. It is difficult to find out exact numbers because the statistical data is not complete. Official statistics have been collected since 1830. Lorey studied the development of mathematics at the universities in the 19th century (see Lorey, 1913, 1916). According to him the first "studiosus mathematicus" was mentioned around 1750. This is odd, because in those days it was not possible to study mathematics. Learning mathematics, namely basic algebra and geometry, was part of the general education in teacher training at the philosophy faculty for future teachers of a "Gymnasium" (i.e., Grammar school). Lectures on higher level mathematics at universities started about 100 years later – as explained below. If mathematicians gave lectures about their own research, they talked about it in a more private atmosphere, the so-called *privatissima*.

Lorey analysed statistical data about those university students who passed the state examination in the subject Mathematics and Natural Science in Prussia in the years 1839-1913. He came to the conclusion that: "We see since 1839 and at the beginning slow and after 1870-1885 rapid increase of numbers up to 160 candidates"[1] (Lorey, 1916, p. 22). In the last years of the 19th century this number was declining to the level of about 1840 (that means roughly 25 per year) and in the first years of the 20th century the number was rising to 300 candidates per year. Lorey reports on the connection of mathematics and natural science as a subject of examinations. Since 1899 chemistry and biology were separate subjects for the state examination. Physics and mathematics were closely connected and not separated in the official statistical data.

It is also difficult to find exact data on the numbers of university teachers of mathematics or mathematicians at universities. Lorey wrote that these professors were often called professor of astronomy or physics. A so-called professor of mathematics came later in the 19th century (see Lorey, 1916, p. 17). We know this fact very well from the biographies of famous mathematicians and scientists I. Many of them made important contributions to mathematics and physics, or astronomy, mechanics, or other subjects. Some were working as engineers, some as teachers at grammar schools, some were working in administration, economics, medicine, and so forth. From earlier times we know that philosophers (like Descartes) or lawyers (like Fermat) or others with several jobs (like Leibniz) worked as mathematicians, too. Hilbert gave lectures in physics and mechanics. The pure mathematician or the applied mathematician who specialized in a small part of mathematics as a full time job came later.

Looking back at the 19th century two tendencies can be recognized: On the one hand we have more and more – but all together very few - mathematicians (i.e., researchers at universities and outside universities) and, on the other hand, we have an increasing number of persons who were concentrating on the part of science we call mathematics today. An important number of research results or published papers on mathematics were produced by mathematicians outside universities. Weierstraß, for example, was a teacher at a gymnasium (grammar school) for many years and Kronecker did not work as a teacher at a gymnasium or university at all (see Ferber, 1956).

Another important group and resource for the growing mathematical knowledge were assistants and post-graduate students. After long disputes with other faculties, the departments of natural science including mathematics gained the right to promote students. Dissertations (doctoral theses) and later habilitations (postdoctoral theses) became a source of new knowledge and successful research. Some mathematical dissertations and habilitations became milestones for mathematics and for the career of the authors as famous mathematicians.

From today's point of view this structure seems to be very clear. But 200 years ago it was new. The new type of university offered the chance to build groups of mathematicians: a full professor and his assistant professors and a group of students of mathematics working together. In the 19th century we see a development from a single professor and his group to a department of mathematics with several specialized professors and other researchers.

The Development from Single Lectures to a Regular Course of Mathematics – Another Precondition for Mathematics to Become a Social System

In the 18th century lectures on higher mathematics were very rare. Some professors liked to give such lectures about their research and some did not. The research mathematicians were autodidacts or had a mathematician as a private teacher. Today the situation is completely different. Perhaps there are some mathematicians who are autodidacts but the majority of the research mathematicians have studied mathematics at university. They have had to study all the courses prescribed in the curriculum. Graduation, dissertation thesis, postdoctoral activities, habilitation, and an appointment to become professor, are typical steps of a career as a research fellow or teaching mathematician. Some of the steps from the situation in the past to the present situation should be mentioned because they have contributed to mathematics becoming a relatively autonomous social system.

After 1830 a very important development was a new type of university structure called "mathematical seminar". The first seminars were founded in Königsberg (by C.G.J. Jacobi and F. Neumann in 1834; see Koch, 1839, 1840) and later in Berlin by K.H. Schellbach in 1861. What type of structure was a "seminar"? It was a public institution connected to the university. It should give the students of mathematics an orientation for their studies. A seminar's principal purpose was that students could learn enough mathematics during the first few semesters. As members of the seminar the students were guided to become well trained mathematicians and to become able to do their own research. Schubring (1983, p. 23) explains the importance of these seminars for the development of mathematics in three aspects:

- At the beginning of the 19th century students were not used to studying an academic subject like today. Studying rather meant learning to ride, learning to fence and learning upper class behaviour rather than learning mathematics. The seminar was a way to concentrate the activities of students on studying mathematical topics.
- The seminar had a budget to buy books and to offer a place where students had access to mathematical literature. This was much better than the usual equipment of university libraries.
- A seminar was connected with an obligation for professors: They had to give regular lectures for students, not only well-paid "privatissima".

The next development was the transformation of seminars into departments. A department is a part of the university with a budget for positions such as professors or other staff for teaching and research. Whereas a seminar had been an exclusive activity for a few students only, a department is responsible for the teaching of all students of mathematics. This new and greater task created more teaching jobs for mathematicians. Becoming an assistant professor is a good starting point in an academic career. We see a parallel development in the increase of the quality of teacher education and the institutionalisation of careers for mathematicians at the university. People could start to plan a career although such a plan included a lot of risks – as it does today.

The second half of the 19th century was a very productive time for pure mathematics. The training of students of mathematics was limited to becoming teachers at a gymnasium (grammar school); they could not become engineers then. A diploma in mathematics was only established in Germany as late as, 1942; a diploma in technical mathematics about 20 years earlier. This had a striking consequence for the type of mathematics which was taught at university. Only since then has applied mathematics found its way into the curricula beside pure mathematics.

As will be shown later, this led to conflicts with natural scientists. The important social aspect is that before the introduction of applied mathematics into the departments of mathematics the students had been trained in pure mathematics. These mathematicians had fewer problems than applied mathematicians in saying "I'm a mathematician," and not "I'm a natural scientist with major activities in mathematics". This is one root of the separation of mathematics from the natural sciences.

The Development of an (Inter-)National Mathematical Community

As mentioned before, the most important part of a social system "science" is communication. Its existence depends on an optimal infrastructure for communication. It was necessary to build up structures where results of research could be exchanged. In the 19th century an interesting development took place in this area. At the beginning of the 19th century mathematicians and other natural scientists started to found associations; at the end of the century national and international associations of mathematicians were founded separately from natural sciences.

The first organisations were local groups. Lorey reports that the Mathematical Society in Hamburg was founded in 1690 (Lorey, 1916, p. 221). Many other similar societies were founded in different university cities. Lorey describes the path from these local organisations to the foundation of the Societies like the DMV (founded in 1890) as a long process including writing letters, travelling, visiting colleagues, invitations to lectures at other universities or local organisations, and meetings of small groups of mathematicians. One of these meetings was the starting point of the journal *Mathematische Annalen* (Mathematical Annals).

Another point on the way was the separation from other organizations. In 1822 the *Gesellschaft Deutscher Naturforscher und Ärzte* (Society of German Natural Researchers and Medical Doctors) was founded. In the year 1843 a section *Mathematics and Astronomy* was started. During the annual meeting of the *Gesellschaft Deutscher Naturforscher und Ärzte* in Heidelberg the *Deutsche Mathematiker – Vereinigung* (DMV) was founded in 1890 (see Lorey, 1916, p. 213; for more information about the importance and history of the *Gesellschaft Deutscher Naturforscher und Ärzte* see Pfetsch, 1974, & Inhetveen, 1976, pp. 101–110).

Lorey noted some drawbacks on the way to a national organisation. Sometimes it was difficult to convince the mathematicians, some of whom were a little narrow-minded and kept their distance from such national questions. A meeting in a south German university town could not take place because the mathematicians living there had decided to leave the city during the time when the other mathematicians were to come.

Two persons are identified as being responsible for the success of the foundation process: Georg Cantor and Felix Klein. Lorey thinks that Cantor undertook the main efforts. He cites a *Laudatio* held by A. Gutzmer for the celebration of Cantor's 70th birthday (see Lorey, 1916, p. 215). Tobies collected documents that prove that Klein played a main role in the background to the process of the foundation of the *Deutsche Mathematiker-Vereinigung* (see Tobies, 1985, 1989).

Summary of 19th Century Developments

I have summarized some aspects of the development of mathematics in the 19th century to point out how the number of mathematicians was growing and how an effective communication structure was built up. At the end of the 19th century there were several universities with mathematics departments, a growing number of mathematicians working inside and outside of universities, an organisation of mathematicians, and many mathematical journals – some of which were very important. Altogether mathematics took a lot of important steps towards becoming a social subsystem, but it was still an integral part of the social system "natural sciences including mathematics" because these disciplines were not clearly separated.

Now I want to have a look at the discussion on the truth and relevance of mathematical theory. These debates were starting points to a separation, but the separation itself took place later on. Why? Luhmann (1970) says that the main reason for the differentiation of social systems is a different function. This means for scientific systems that there must be a difference in the way the truth of a new theory is proved and accepted. The function of scientific social systems is to separate true and not true theories (whatever "true" means).

Starting Points of the Functional Differentiation

A long time ago the development of mathematics was hampered by discussions on the truth and the meaning of the results. Non-Euclidian geometry is a well known example of this. Euclid from Alexandria (365–300B.C.) wrote the *Elements*, a famous collection of mathematical theorems. He defined what we call *Euclidian Geometry*. The fifth postulate said that in a plane there was one and only one straight line parallel to a given straight line through each point that was not an element of the straight line. Over more than 2000 years many mathematicians have tried to prove that this postulate can be deduced from the others, but they have not been successful. They only discovered other equivalent postulates.

Kant was convinced that the Euclidian geometry was the only possible geometry (see Kant, 1781). His philosophy influenced the scientists greatly; nobody published results on other types of geometry.

Gauss founded a new approach to geometry. He asked formally: What would happen, if there were more than one parallel straight line? Would it be possible to define a postulate system with this postulate, but without contradictions? About 1790 he found the non-Euclidian hyperbolic geometry but did not publish his results. He wrote letters to other scientists like Schumacher or Bessel (see Wußing, 1983) to inform them about his new ideas on geometry and his fear of publishing something contradictory to Kant's philosophy. Two mathematicians published papers on non-Euclidian geometry (Bolyai in 1832 and Lobatschewski in 1826), but both had problems in gaining acceptance. The latter wrote that the new geometry did not exist in nature but in the human mind only. Maybe it was not useful for measurement in nature, but it would open a wide area of application in both geometry and analysis (see Lobatschewski, 1899, p. 83). From today's point of view this typical formalistic argument is a common reason to publish results, but 200 years ago this was not accepted. Mathematics was located in philosophy faculties and could not act autonomously in decisions about the truth of mathematical theories. Some months before Gauss died (June 10^{th} 1854), Riemann presented the results of his research on geometry – the theory of the *Riemann'schen Mannigfaltigkeiten* – to become a teacher at the university level. He presented many types of non-Euclidian (elliptic) geometry and an explicit rebuttal of Kant's theory on geometry.

In the 19^{th} century a lot of work was done to find out more about analysis, especially about proofs and the foundations of analysis. Looking at the history of analysis we find some significant works written by Cauchy (1821), and about 50 years later by Weierstraß, Dedekind, and others. One basic idea was a foundation of analysis that had a quality similar to other parts of mathematics, especially arithmetic (see Cauchy, 1821; Klein, 1926; Spalt, 1981; Struik, 1976). An unexpected result of the use of arithmetic was a formalized approach, the possibility of looking at analysis from a formal point of view. Du Bois-Reymond published an example for a steady function that cannot be differentiated. He called the function *disconcerted* (Du Bois-Reymond, 1875, p. 21). Some of his colleagues called such functions *pathological*, but this expression was not accepted by many mathematicians. From my point of view this shows quite well that a formal approach often leads to unexpected results. Although the existence could be formally proved, nobody could imagine such functions or see them in reality. Looking for examples in reality is a typical method of natural scientists; today formal working mathematicians do not use this method. Their argument is that the future may show if the new theory is useful in reality. If it is not, this does not matter for acceptability today or in the future.
Cantor's results were not only "disconcerted" or "pathological". He asked for the potency of the set of natural and real numbers and showed by using formal methods that there were at least two types of infinity. Firstly, the *Kardinalzahl* (i.e.,cardinal number) of □ and *R*. Following his definition, the cardinal number of *N* is aleph 0 and the cardinal number of *R* is aleph 1 (see Cantor, 1874, republished in 1932 – his paper was published in Crelle's Journal). How are they connected? *R* is equivalent to P(N), the power set of *N*. Cantor's next question was simple: What is P(R) ? It is a set with the cardinal number aleph 2. And many questions later: Which set has the cardinal number aleph Omega that is bigger than aleph n for all n (n element of *N*)?

This was too much for his colleagues – they did not accept it. But a generation later Hilbert (1923) celebrated Cantor as the creator of a paradise for mathematicians. What had caused this change? Wang sees “The main reason, why Cantor has been so much more influential is probably his ability to renounce applications and develop the theory of sets more and more for its own sake. By generalizing and following up logical conclusions, Cantor became the founder of set theory” (Wang, 1954, p. 244).

Let me add a little *pedagogical remark* to the historical debates about geometry and analysis in the 19th century and to similar debates on research results by Graßmann (see: http://www.maths.utas.edu.au/People/dfs/Papers/GrassmannLinAlgpaper/GrassmannLinAlgpaper.html) and Hamilton (1853) – two prominent examples for the change of view in history: I think it would be a very good idea to read the original papers to understand these debates. It would open a new and deeper view inside mathematics. Most mathematicians today do not know that there was any debate in history on the acceptance of well-proven research results. Mathematicians and especially mathematics teachers should know more about mathematics!

The Crisis of Foundation of Mathematics at the Beginning of the 20th Century

Cantor and the definition of a set is a key to the next step, the historical crisis of mathematics about a hundred years ago. I will show that the most important result of this crisis was the birth of mathematics as a separate social system with a separate function: Mathematicians defined their own way of deciding the truth of mathematical research results.

Today we have a very useful synthesis of works about sets and formal logic. Papers written in English use some hundred different words and the well-known symbols of sets and logic. Mathematicians all over the world can read these papers if the words and symbols are used correctly. Thus mathematics has developed a unique international communication code. If we compare this communication code with other scientific disciplines we see enormous differences. For example books about philosophy use several thousands of non-standardised words and even some philosophers point out that it is difficult to understand what Hegel exactly meant when he used his special terms.

This useful international communication code has but one disadvantage: it is an abstract code. Applying it means filling it with sense for the area where mathematics should be used. This is not simple.

The development of this communication code was not simple, either. About 130 years ago Cantor wrote the basic ideas of the set theory. 30 years later Russell published his book *Principles of Mathematics* (Russell, 1903) including the proof for the inconsistency of the basic definition of a set. He defined the set M of all sets that do not include themselves. This definition leads to a contradiction: M cannot be an element and not an element of M, but M is a set according to Cantor’s definition of a set. This antinomy had been found several years before by Burali-Forti (1897), but Russell’s book had more readers.

At the beginning of the 20th century many mathematicians used the symbols of set theory as a useful language to write down their results. Therefore many of them wanted to know: Is this problem important for my results? Are they correct? If we are using a wrong precondition, they may be incorrect, because from false proposals everything can be deduced.

In this situation it was very helpful that two technical solutions were found and offered by Zermelo (1908) and Fraenkel (1927), and by Bernays (1930 – 1931), Gödel (1931), and von Neumann (1931). Both solutions avoid the antinomy but are formal. The authors define sets in an abstract way without any relation to reality or to philosophical reason.

From different philosophical positions several attempts were made to solve the crisis of the foundation: The existence of the mathematical objects and the proof of the truth of mathematical theories were the main questions in the philosophical debate. I can’t report this debate here (see Maasz, 1988), but I want to outline some important positions: the logicism, formalism and intuitionism.

The Logical Proposal for a Solution and its Problems

The first attempt to solve philosophical problems was made by mathematicians who were known as experts in mathematical logic like Frege, Russell, and Whitehead. Frege showed that the problems of basic definitions of a set by Cantor are language problems. Words of different "types" were mixed. (For details see the explanation of types by Frege in his theory of types – Frege, 1884, 1893, 1903).

For a long time logic was a part of philosophy. If we look at important developments we see Aristotle as a starting point, the rules of argumentation in the Middle Ages (Syllogism) and a new type of thinking about logic in the 19th century. Russell (1901) wrote that pure mathematics was invented by Boole's book on *Laws of Thought* (1854). Frege's logical foundations of mathematics were much stricter and clearer then everything written before (see Thiel, 1972, p. 93). But when Russell found a contradiction and told Frege about it in, 1902 – a short time before the second issue of his book should be published – this became a problem for Frege. He tried to find a way out but was not successful (see Quine, 1955, for details). Fraenkel (1927) and Hilbert (1925) report that this made him resign.

The next attempt by Russell and Whitehead avoided antinomies in their *Principia Mathematica* (Whitehead & Russell, 1910, 1912, 1913). They invented a more sophisticated theory of types but they needed two basic axioms: infinity – the existence of an unlimited totality is postulated – and reducibility: "every function of one variable is with all its values equivalent to a predicative function of the same argument" (Whitehead & Russell, 1910, p. 166) These axioms postulate what should be proved, namely the existence. In the preface to the second edition Whitehead wrote: "This axiom has a purely pragmatic justification: it leads to the desired results, and to no others. But clearly it is not the sort of axiom with which we can be satisfied" (Whitehead, Russell, 1925, p. xiv). If we assume what we want to prove we are able to prove everything. So this was a very useful and honourable work (see Rusawin, 1968, p. 243) but not the philosophical foundation of mathematics the researchers were looking for.

The Intuitionistic Proposal for the Philosophical Foundation of Mathematics

Since Plato and Aristotle we have had an ongoing debate as to whether mathematical objects are created or found by mankind (see Becker, 1927, p. 572). From the Platonist point of view mathematics exists independently from us in the area of ideas. Our task is to discover it like a sailor discovers new islands or even a continent in the ocean (see Stegmüller, 1978, p. 675). Intuitionism is a part of Constructivism, as explained in books of Aristotle, Kant, and others. Aristotle, however, believed that mathematical objects are constructed by human beings through abstraction. He collected arguments against Plato's assumption in the first, sixth and twelfth book of his "Metaphysics" (see *Organon*, 1948).

The intuitionistic world centres around the concept of "intuition": This means an act of gaining knowledge inside one's own mind. Thus intuition is not a result of social or empirical activity but a result of thinking – a priori intuition – (see Brouwer, 1907, p. 179, or Heyting, 1934, p. 3). This is not to be confused with the Russian understanding of intuitionism. Markov put it this way: "I can't accept that 'intuitive clearness' is a criterion for truth in mathematics because this means the total triumph for subjectivism in mathematics. This is not in accordance with the opinion that scientific research is a form of social activity" (cited in Rusawin, 1968, p. 262). The debate about the social basis of scientific constructions continues until this day.

What did intuitionism want? To save Cantor's set theory was not among its principal targets. The intuitionists like Brouwer, Heyting or Weyl wanted to find an epistemological basis for mathematics. The method proposed was to exclude the actual infinity and the method of constructing everything. The Platonist deduction from "it is formally correctly defined" to "it exists" should not be allowed.

The result of the intuitionistic work was a new type of mathematics. It was philosophically well founded but it had additional rules and was less extended. The majority of mathematicians did not want to accept this. Hilbert wrote that the intuitionists had curtailed mathematics and had restricted research in this direction. He wrote: “Mathematics is in danger of losing a big part of its most valuable treasures” (Hilbert, 1922, p. 159). Bernays argued: “It is an unreasonable demand from philosophy to mathematics to abandon a simple and more efficient method”, (Bernays, 1930–1931, p. 351). This argument shows exactly what the theory of a social system would predict: The better functionality is a main reason for opposing intuitionism. As part of a bigger social system mathematicians had to make allowances for philosophy and its demands on its methods. As a separate social system mathematics would be relatively autonomous and free to define its own methods and criteria for truth.

The intuitionistic demands were rejected. Only very few mathematicians worked on this basis. In the second half of the 20th century a group of scientists located in Erlangen (Germany) renewed the intuitionist argumentation. They showed that it was possible to prove all parts of analysis that are needed for applications in the real world (see Lorenzen, 1958; Lorenzen & Schwemmer, 1972).

The Way from the Formalistic Proposal for a Philosophical Basis by Hilbert to the Formal–Axiomatic Solution

The formalistic position consists of a variety of different approaches. Most of them are based on a Platonist point of view. Hilbert was one of the prominent representatives of formalism though he was not a pure formalist. He always asked for reasons for the mathematical theories and for their connection to reality – like in physics. Bourbaki (1971) characterizes Hilbert as an atypical formalist. According to Bourbaki, formalists believe that a formalized language has the sole function of being an unambiguous vehicle for communication. It does not matter for them whether the mathematical object exists in reality or not, as long as it is possible to write down the knowledge about it in a formalized language. But Hilbert always has believed in an objective mathematical ‘truth’ (see Bourbaki, 1971, p. 48).

Hilbert’s aim was to definitely remove all doubts as to the certainty of mathematical conclusions (see Hilbert, 1923, p. 178). In order to reach this goal, he developed an axiomatic method. His axioms are, however, different from those of Euclid. Euclid had defined axioms in a way that they are clear and evident statements about the truth. Hilbert wilfully relinquished a relation to reality, as can be read in his book on geometry (see Hilbert, 1899). In a letter to Frege dated December 29, 1899, he explained: “If I imagine any system of objects like the system of love, law or chimney sweeper and take my axioms as relations between these objects my conclusions (for example, Pythagoras) are correct for these objects.” (See also Steck, 1941).

This is a good illustration of formal thinking. Hilbert proposed that mathematicians should work with formal theories. The connection of these formal theories to reality was an additional task. This happens when the formal theory is applied in a special context or when it is filled with facts (numbers, data) from reality. If a mathematical theory presupposes formal axioms, all consecutively deduced theories are formal. All of mathematics becomes a formal theory (see Hilbert, 1925, p. 177). Mathematics becomes a collection of formally proved or deduced theories. The relation to reality is not clear and open. Hilbert did not want to stop at this level. He developed a second part of mathematics called the theory of proofs. He called this part of mathematics “Metamathematics”. Metamathematics works with the proof of the other part of mathematics as objects to make sure that the axioms are free of contradictions (see Hilbert, 1923, p. 180).

From a philosophical point of view it is not sufficient to prove that axioms are free of contradictions. A proof doesn’t show the existence of the axioms – if you are not a convinced Platonist. However, independent from the philosophical debate, Hilbert’s program did not work. Gödel showed “that Hilbert's program is essentially hopeless” (von Neumann, 1947).

From my point of view, Hilbert’s program was very successful in a non-intentional aspect – the division of mathematical work into normal work (i.e., formal work) and thinking about its foundation. Most of the mathematicians did not think about philosophical problems and foundations any more. They used the formal approach to work faster and more efficiently. Only very few mathematicians decided to do research on logical, philosophical or basic problems. For mathematics as a whole this division of work was a good method to be more successful. As is well known from other areas, the division of labour enhances the chances of increasing the output.

The Last Step on the Way to "Mathematics as a Social System": The Bourbaki Group

In 1934 a group of famous mathematicians planned to give a complete description of all parts of mathematics based on a formal axiomatic method, assuming nothing but formal axioms and deducing the rest (see Bourbaki, 1934, 1971). Cartan, one of the members of this group, wrote that they were convinced that such a building up of mathematics "ex nihilo" should be possible (Cartan, 1959, pp. 8, 12).

The Bourbaki group was very successful. Some claim that the name had been taken from a general of Napoleon's army, others maintain that the letters of the name represented members of the group. Their books gave mathematics a new basis and a very useful communication code. The main reason for the success of their work is the standardized way by which they described all aspects of mathematics. It gave an new and better overview than the "Enzyklopädie der Mathematik" which had started about 50 years earlier but was never finished (see Maasz, 1987).

Bourbaki's books opened up a wide area of new questions for mathematicians because this new view on all aspects of mathematics implied many hidden relations of different parts of mathematics. A formal view motivates one to ask more formally : "What happens if I take more dimensions or if I apply a result from one part of mathematics to another part of mathematics?". This research was not slowed down by any questions about meaning or about the connection to reality or by the need to prove the possibility of application. This formal–axiomatic method was exactly the type of method which Luhmann (1984) has described as the ideal communication code.

The Bourbaki group has clearly pointed out, that they have not been interested in philosophical questions. I do not want to argue with them about this point! (See Bourbaki, 1971, p. 39). From a philosophical point of view the philosophical problems have not been solved but suppressed (see Thiel, 1972, p. 127). I would like to add that the statement as to not being interested in philosophy is not equivalent to the absence of a philosophical position. Bourbaki is based on Plato, and Platonists believed that exists what can be defined.

Conclusion

The agreement of the majority of mathematicians to use the formal–axiomatic method proposed and introduced by the Bourbaki group was the agreement to make mathematics a separate social subsystem. This method defines a function of the subsystem "mathematics" that fulfils the criteria Luhmann (1984) explains. The consequence was a functional differentiation, a separation from the other sciences.

As shown in this article, the foundations for mathematics to become a subsystem of the science system were laid in the 19th century. The agreement on a specific mathematical method to decide the truth of theories was the birth of the social subsystem called "mathematics".

Finally, the implications for educators of adults learning mathematics are that they should know more about the history of mathematics and, from this article, keep in mind that the development of mathematics is strongly influenced by decisions made by mathematicians. Following the interpretation guided by Luhmann's theory, it is significant that these decisions are made according to what "should" happen.

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Folding Back and the Growth of Mathematical Understanding in Workplace Training

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Abstract

This paper presents some initial findings from a multi-year project that is exploring the growth of mathematical understanding in a variety of construction trades training programs.[1] In this paper, we focus on John, an entry-level plumbing trainee. We explore his understandings for multiplication, fractions and units of imperial measure as he attempts to solve a pipefitting problem. We consider the apparently limited nature of his images for these concepts and the role of 'folding back' in enabling his growth of understanding. We contend that it cannot be assumed that the images held by adult apprentices for basic mathematical concepts are flexible, deep, or useful in specific workplace contexts. We suggest that folding back to modify or make new images as needed in particular contexts is an essential element in facilitating the growth of mathematical understanding in workplace training, but also offer a note of caution about ensuring that this is genuinely effective in its purpose.

Mathematical Understanding and Workplace Training

Although recent years have seen an increase in the attention paid by researchers to mathematics in the workplace there is still only a limited body of work that considers cognition and understanding in a vocational setting. Research that does exist in the area of adult mathematical thinking within a vocational setting is primarily concerned with the use of informal mathematics (e.g., Noss, Hoyles, & Pozzi, 2000), or the place of mathematics within work (see Bessot, 2000a, 2000b; Eberhard, 2000; Harris, 1991; Millroy, 1992; Scribner, 1984a, 1984b, 1984c; Smith, 1999a, 1999b; Wedege 2000a, 2000b), rather than with the process of coming to understand mathematical concepts for the individual learner in the workplace.

Straesser (2000a) in his discussion of workplace vocational mathematics instruction noted two distinct kinds of pedagogies, the first being "modelling" and the second being "legitimate peripheral participation" (see also Lave & Wenger, 1991). He notes that the first is usually employed in a classroom setting, where:

> the situation is to come from the workplace, the mathematical model rests upon mathematical structures and algorithms known before or taught on the spot and the solution of the model hopefully can be interpreted in a way to cope with the given professional situation (p. 70).

In contrast to this, legitimate peripheral participation relates to training on the job, where "learning takes place at the workplace whenever it is needed by the workplace practice and its problems" (p.70). In this study we are interested in the first of these two learning situations, that of the workplace training classroom, where apprentices engage with contexts and problems from the workplace, but which require mathematical models in their solution. As Straesser (2000a) states "in most cases, modelling vocational problems by applying mathematics is a major difficulty for the future worker – especially the extraction of the mathematical model from a professional situation at hand" (p.70). He does not, however, offer any detailed analyses of the kinds of difficulties faced by apprentices in extracting the mathematics and working with it, nor consider the ways in which they go about developing and using appropriate mathematical models.

Pozzi, Noss, and Hoyles (1998) in discussing the use of artefacts and tools in the workplace note that the need for workers to understand "the models which underlie their artefacts, to sort out what is happening or what has gone amiss … typically occurs when there is a breakdown, and in such a situation, people need to represent to themselves how the underlying structures work" (p.118). They also suggest that modelling is "a dynamic process of building connections between a situation and its mathematisation, rather than a process of decontextualisation" (p.119). We consider the growth of mathematical understanding in workplace training as a dynamical process, and in this paper focus in detail on the way that one apprentice, when in difficulty, attempts to understand the mathematical concepts required by the task and those embedded in the use of a measuring tape.

The nature of mathematical understanding

The research reported in this paper is framed by the Pirie-Kieren theory for the dynamical growth of mathematical understanding (Pirie & Kieren, 1989, 1992, 1994). This theory views understanding not as a static product, nor as something to be acquired and then applied, but instead it is characterised as occurring in action and thus constantly evolving. Therefore, rather than being seen as a linear process, the growth of mathematical understanding is characterised as:

> not simply a matter of acting in abstract ways with more and more abstract mathematical objects. Such growth in fact entails a dynamic and a connection between more and less formal, abstract and sophisticated activities. Because growth in understanding in action occurs in contexts, a study of the growth of understanding must necessarily take into account the interactions that a person has with and in such contexts, including interactions with materials, other students and teachers. (Kieren, Pirie, & Gordon-Calvert, 1999, p. 229)

We agree that such a view of the nature of mathematical understanding is a vital one for the workplace training setting, where understandings are continually situated in specific problem-solving contexts, being dependant both on specific materials and on interactions with other workers and trainees.

This location of understanding in the "realm of interaction rather than subjective interpretation" and a recognition that "understandings are enacted in our moment-to-moment, setting-to-setting movement" (Davis, 1996, p.200) allows and requires the discussion of understanding not as a state to be achieved but as a continuously unfolding phenomenon. Hence, it becomes appropriate not to talk about 'understanding' as such, but about the process of coming to understand, about the ways that mathematical understanding shifts, develops and grows as a learner moves within the world.

The Pirie-Kieren theory provides a way of considering the socio-cultural environment of the learner, through seeing the individual not merely as existing within a particular context or discourse (as suggested by Lave, 1996; Walkerdine, 1988) but as co-constituting, co-existing and co-emerging with the context. As Davis (1996) writes 'the world's relationship to the organism is not merely uni-directional and constraining; the organism also initiates or contributes to the enactment of its environment" (p.10), and to re-emphasise the notion of embodied action, "our sensorimotor capacities are embedded in and continuously shaped by broad biological, social, and historical contexts" (p.11). Thus, this notion of understanding, rooted in an enactivist position, acknowledges the concerns of those working from a socio-cultural perspective, and does not dispute the significance of context in considering learning. Indeed Davis (1995) states:

> enactivism does not dismiss the varied critiques of mathematics education. Rather, the framework offers a means of incorporating cultural commentary with discussions of individual cognition. It does so by arguing there is a certain self-similarity between processes of individual cognition and of collective action. (p. 8)

The theory thus offers a way to account for the growth of personal dynamical mathematical understanding in a very detailed way, whilst still recognising the vital role of the context in which this growth is occuring.

The Pirie-Kieren theory posits eight layers of understanding together with the cognitive activity of 'folding back' as crucial to the growth of understanding. A diagrammatic representation of the theory is provided in figure 1. The theory and the model can be used to "capture growth in mathematical understanding as *pathways of understanding* that are unique to persons and topics and co-emerge with the particular situations in which they find themselves" and also "to observe activity over many time-scales" (Kieren, Pirie, & Gordon-Calvert, 1999, p.219). It is thus a tool that offers the user a means to observe and describe the growth of mathematical understanding for a chosen learner, for a chosen mathematical concept, over a chosen period of time. These choices are made by the observer, and are driven by their particular focus of interest.

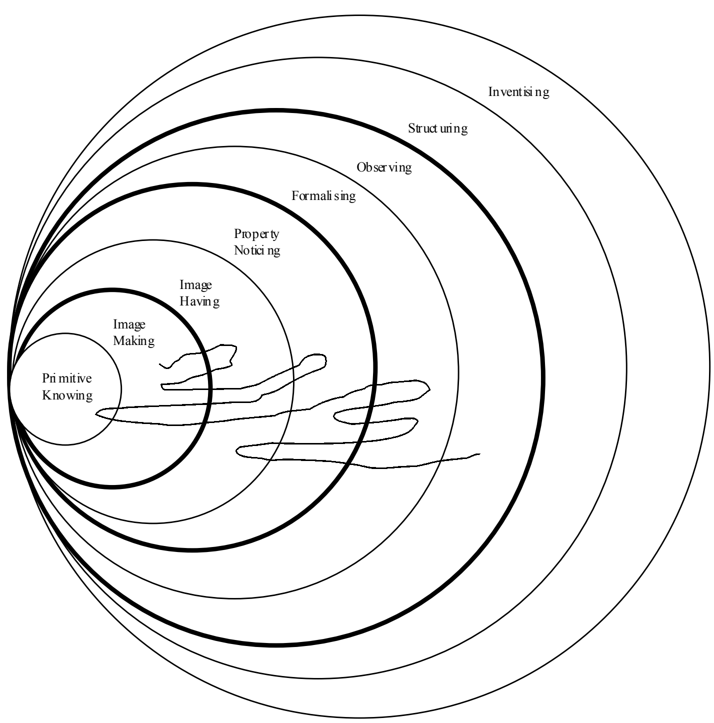

Figure 1. The Pirie-Kieren model for the Dynamical Growth of Mathematical Understanding.

Two of the inner layers are defined as *Image Making* and *Image Having*, and it is these layers that are relevant to our discussion in this paper. At Image Making learners are engaging in specific activities aimed at helping them to develop particular ideas and images for a concept. By “images” the theory means any ideas the learner may have about the topic, any “mental” representations, not just visual or pictorial ones. Image Making often involves the drawing of diagrams, working through specific examples or playing with numbers. However, it does not have to have an observable physical manifestation, it is the thinking and acting around the concept that is the actual process of making an image.

By the Image Having stage the learner is no longer tied to actual activities, they are now able to carry with them a general *mental* plan for these specific activities and use it accordingly. This frees the mathematical activity of the learner from the need for particular actions or examples. At this layer, the learner *has* an understanding, although this may still be very specific, mathematically limiting and context dependent, which they are able to employ when working on mathematical tasks.

The notion of folding back

Although the layers of the Pirie-Kieren theory develop from the concrete to the more abstract, or from the specific to the general, it is important to recognise that the growth of understanding does not occur through a simple linear pathway through the layers. Instead, mathematical understanding is seen to emerge through the continual movement back and forth through the layers of knowing, as individuals reflect on and reconstruct their current understandings (See figure 1 for a representation of a possible hypothetical pathway of growth).

A key feature of the theory is the idea that a person functioning at an outer layer of understanding, when faced with a problem that is not immediately solvable, needs to return to an inner layer of understanding to examine and modify their existing ideas and thinking about the concept. This process is known as ‘folding back’ implying that when a learner revisits earlier images and understandings for a concept he or she carries with them the demands of the new situation and uses these to inform their new thinking at the inner layer, leading to what may be termed a ‘thicker’ understanding for the concept.

As suggested, folding back occurs with a purpose, namely to extend one’s existing understandings which have proved to be inadequate for handling a newly encountered problem. It is in response to an obstacle that the learner re-visits earlier understandings, aiming to modify, collect, or build anew conceptions which will allow the difficulty to be overcome through an extended understanding of the topic.[2] Hence the inner layer activity is informed by what the learner already knows and by what they need to be able to do.

It is the fact that the outer layer understandings are available to support and inform the inner layer actions which gives rise to the metaphor of *folding* and *thickening*. Although a learner may well fold back and be acting in a less formal, more specific way, these inner layer actions are not identical to those performed previously. Folding back can be visualised as the folding of a sheet of paper in which a thicker piece is created through the action of folding one part of the sheet onto the other. The learner has a different set of structures, a changed and changing understanding of the concept, and this extended understanding acts to inform subsequent inner layer actions.

Folding back then is a metaphor for one of the processes through which understanding is observed to grow and through which the learner builds and acts in an ever-changing mathematical world. Folding back accounts for and legitimates a return to localised and unformulated actions and understandings in response to and because of this changing world. For example, folding back to the layer of Image Making might again involve such physical actions as drawing diagrams, working with manipulatives or playing around with numbers. However, it should be noted that:

> such image-extending inner-layer activity to which a person returns is reconstructive (or recollecting) in nature. It is not simply the instant recall of a known piece of information or action

> sequence. It involves remembering or recombining or extending actions, images, formalizations or theorems. (Kieren, Pirie, & Gordon-Calvert, 1999, p. 218)

Whilst it might be initially be seen as a step backwards in terms of the observable actions of the learner, folding back is likely to lead to a thicker and deeper understanding of the mathematical concept, and is an essential stage in the dynamical growth of mathematical understanding.

Images and Folding Back in Workplace Training

Apprentices in workplace training are often re-learning mathematics that they have already met and have existing images and understandings for, whether these are ones developed in school or elsewhere and whether formal or informal in nature. As they engage in mathematical activity during training apprentices may need to re-visit these understandings and images, make sense of them again in specific trades situated applications, and if necessary construct new understandings which will allow the mathematics to be useful in the new context.

In the data drawn on in this study, the mathematics is presented within a problem context, where the production of a physical solution is required – that of a piece of pipe cut to a correct length. To be able to apply appropriate mathematics and perform suitable calculations is something of a secondary aim, what matters in the workplace is the resulting product of the mathematics. This is in contrast to the purpose of the same problem were it to be posed in the school classroom. Here, it would be the mathematics that would likely be the focus, the problem merely being a context in which to set this. The conclusion to the problem would likely be a calculated answer, rather than a piece of pipe suitable for the task in hand. Although the mathematics required in both situations is quite possibly the same or similar, it is the application (and therefore the understanding) of this mathematics that differs in the two contexts. Also, of course there are very different consequences for the correctness of the answer. In the school classroom, an incorrect answer will likely result in nothing more than a mark on a piece of paper, whereas in the workplace there are real costs associated with such errors. Thus different images may be required for the same concept when used in the workplace rather than the school classroom (see Forman & Steen, 2000), or a previously held image may need to be modified or broadened.

For example, in the elementary school fractions are often taught with reference to parts of a whole circle (usually described as being a pizza or pie) and the image here would be quite specific and one based on an area representation. However, when using fractions in the context of measurement, it is more appropriate to see a fraction as a point on the number line – although of course to be able to read a fractional unit of measurement still requires an understanding of the part-whole relationship. This number-line image is particularly important for working with measurements in imperial units, where lengths are stated in fractional units, unlike in metric units where decimals are more commonly utilised. (For example, one would rarely talk about three and seven tenth centimetres, though it is of course mathematically valid and correct). As Martin, Pirie, and Kieren (1994) note:

> fraction learning involves constructing an ever more elaborate, complex, broad and sophisticated fraction world and developing the capacity to function in more complex and sophisticated ways within it. Such an achievement will prove impossible if the foundations laid by the images the learners hold are not adequate to the task (p. 248).

However, it is important to recognise that merely engaging in an appropriate act of folding back (e.g. to make a new image for fractional units) does not automatically guarantee that the learner will be able to immediately overcome the problem which prompted the invocative shift. For a learner who is able to use their extended understanding to overcome the original obstacle we term such folding back ‘effective’. It is important to note that this does not imply that the learner now has a complete solution and acknowledge that further folding back may be required before a sufficiently extended understanding exists. The key feature of effective folding back is that the learner is able to return to the outer layer and apply a newly extended understanding to the original problem in a useful way. Continued working may yield another new and different obstacle for the learner, necessitating further back and forth movement, but this is distinctly different from being unable to make use of the new constructs at the outer layer. In the former case the understanding of the learner is still growing through a continual back and forth movement, whilst in the latter it has been temporarily halted and is termed 'ineffective'.

Methods and Data Sources

The larger study, currently underway, is made up of a series of case studies of apprentices training towards qualification in various construction trades in British Columbia, Canada. This paper presents some initial findings and discussion that draws on one of these case studies, and although our conclusions are specific to this case, we would suggest that there are implications that may be relevant to other areas of workplace training and the use of essential skills. The trainees and their instructor were observed and video-recorded over a number of sessions. The episode on which this paper focuses involved a small group of apprentices in the shop working to calculate the length of a pipe component required for a threaded pipe and fitting assembly to be built to given specifications. This followed a formal lesson on this procedure in the classroom. The second author acted as a participant observer in this session and worked closely with individual trainees as they engaged with the task. The video recording of this episode was analysed using the Pirie-Kieren theory with a particular focus on identifying the mathematical images held, accessed, made, modified and worked with by John as he engaged with the task. It should be noted that the transcript offered below represents a very short extract from a number of hours of taping, and some of the comments and conclusions we offer draw on data beyond that presented here.

John and the pipefitting task

We now move to consider John's understanding of mathematical concepts such as fractions, measurement and number and use elements of the Pirie-Kieren theory to describe the way that he is seen to engage in acts of folding back as he works on making appropriate images for these concepts.

A drawing of the pipe assembly to be constructed is shown in figure 2. The apprentices were assigned the task of constructing this assembly with a *centre-to-centre measure* (C-C) of ten inches. There are two ways to approach the calculation of length of the straight pipe (P) required to join the two fittings to meet the given specification:

Method one: Pipe length (P) = the centre-to-centre measure (C-C) minus the *take-off,* where the *take-off* = two times the *fitting allowance* (A) minus two times the *thread makeup* (E).

Method two: Pipe length (P) = the centre to centre measure (C-C) minus two times the fitting allowance (A) plus two times the thread makeup (E).

The values for the fitting allowance (A) and thread makeup (E) were provided elsewhere.

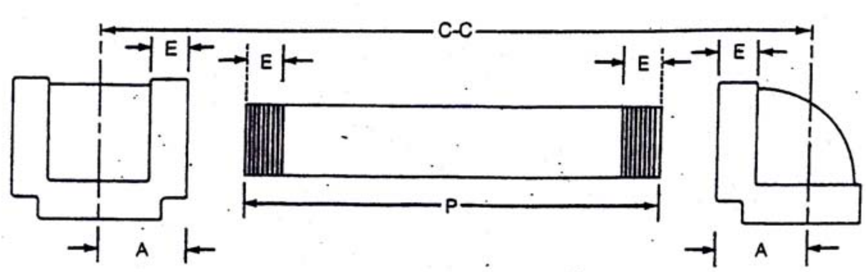

Figure 2. Pipe assembly to be constructed by John using standard pipe fittings and a cut piece of pipe.

Episode 1: Knowing How Without Knowing Why – Ineffective Folding Back

The following episode begins at a workbench in the shop as John works to make sense of the calculations needed for this task.
(John and Researcher)

J: Ok. So what I do now is, I know that it's ten, what I've got to have total.
R: Yeah.
J: I got to multiply the take-off twice, because on each end, right? *(Here John uses the term 'take-off" incorrectly to refer to fitting allowance (A))*
R: Yeah.
J: So what's got me.
R: So you multiply one and three quarters twice?
J: Yeah. One and three quarters gives me three and one sixteenths. Right?

In this first extract John has recognised the need to obtain the total *take-off* amount to accommodate fittings to be attached at each end of a pipe to make a ten-inch centre-to-centre pipe assembly. He has obtained the correct measurement for the *fitting allowance* (1 3/4") from an industry standard reference table for his calculation and written down:
1 3/4 x 2 =
on his sheet of paper. He has an understanding of what is involved in solving the problem, that he needs to remove a length of pipe from each end, and knows that he now needs to translate this understanding into a numerical calculation that can be carried out on a calculator. Once John has translated his understanding of the fitting allowance for both ends into an appropriate mathematical calculation, he uses his calculator to find this product. He enters: $1\frac{3}{4}\times 1\frac{3}{4}=$ on his calculator a number of times while he works on the problem, getting an answer of $3\frac{1}{16}$ each time. While we cannot say with certainty why he chose to perform the calculator operations that he did, we would suggest that one factor is the mathematically ambiguous way (from our perspective) that he frames the required operation for himself, and the image, or lack of image, which underlies this.

John says, "I got to multiply the take-off twice, because on each end, right?" Here he is shifting from his appropriate (non-mathematical) pictorial and physical image for the problem – of an amount to be taken off each end – to one that is exclusively symbolic or numeric in form. However, in using language to re-formulate his understanding symbolically, he states that you have to "multiply the take-off twice" which easily lends itself to a symbolic representation of "$1\frac{3}{4}\times 1\frac{3}{4}$". It would seem that even although at one point he wrote $1\frac{3}{4}\times 2$, he may have been reading this as "one and three quarters multiplied by itself", seeing the symbols as a representation of the visual problem rather than as a calculation to be performed.

We contend that John's difficulty lies in the images that he has for the mathematical concepts being used here, especially those for multiplication. In trying to solve the problem, we see John having a viable visual and physical image of what is required for the task, and then needing to find or construct an appropriate mathematical model. To do this, he needs to have an image for multiplication, specifically one that recognises the idea that "when you multiply something by a natural number, it is the same as adding it to itself that many times." Having such an image would have allowed him to choose the correct calculation to then apply to the question at hand. We do not see John having an image for multiplication as "repeated addition" he does not access this here, nor does he generate it from his pictorial image for the problem, nor is it indeed implicit in his choice of operation.

John does not see the problem in terms of "putting" the two take-off lengths together to produce a single piece to be cut from the pipe, something that would lend itself to thinking in terms of an addition sum rather than a multiplication. We suggest there is a need for John to fold back to Image Making and through performing some actions at this layer, to thicken his image for multiplication in such a way as to allow him to apply this to the new context. This image making might include working with similar examples using whole numbers exclusively, prior to considering fractions, or working with grouping sets of objects. What is interesting here is that John is not convinced that his answer is correct, and he expresses this concern to the researcher, who then works with John on the problem.

R: Let's pull out a ruler. Here. Show me an inch and three quarters. *(John pulls out his tape measure)*
J: Ok.
R: Just show me with your finger.
J: That's an inch. That's my inch and a half. That's my inch and three quarter right there. Am I not correct? (*John indicates the correct points on the tape)*
R: Just put your finger there so I can see it. Ok. So there's your inch and three quarters right there. Add an inch and three quarters to that. And I'd go one step at a time. Like add an inch, and then add another three quarters.
J: Ok, so, I go, I've got an inch and three quarters right here. Which is right here. *(Pointing to $1\frac{3}{4}$ point on tape measure)* So to add another an inch, and another three-quarters to it?
R: Yeah.
J: *(pause)* Ok, hold on. Right here *(pointing at tape measure with pencil)*
R: Yeah.
J: *(long pause)* That would be three right here, right? No. That's one?
R: Right. That's one and three quarters, clearly. That's an inch and three quarters. Now if you add another inch and three quarters to that?
J: Ok. I'm stumped.
R: Ok.
J: I'm stumped. It's simple math here, needed. That's all. I'm not doing it.
R: Give me your right hand. Replace the finger on the tape measure. If that's an inch and three quarters *(indicating this point on the tape measure)* then another inch would be to?
J: Add?
R: Another inch would be to where?
J: Well wouldn't it be to here?
R: To there.
J: Right.
R: And now add three quarters. *(Indicating intervals on the tape measure)* One quarter, two quarters, three quarters. How much?
J: It would be three and a half.
R: Yeah. *(Then a long pause, with no response from John)*

Here we see the researcher taking John out of the context of the problem, and working with him to explore multiplication as repeated addition. The researcher has engaged John in a folding back activity, returning him to the Image Making layer. The hope is that through using a tape measure, a familiar and readily accessible workplace tool, John will make an image for multiplication as repeated addition through engaging with an activity of adding two equal quantities together on the tape measure. This is of course a useful mathematical image for the physical actions involved in cutting the required pipe length, as it encapsulates the idea of putting the two pieces of pipe together, finding the total and then making one cut, on one end, to remove this amount of waste. Although John engages with the activity he struggles with it, and clearly is not sure how to count-on using the scale on the rule, though at the end of the extract he seems to have performed the required addition act.

More significantly though, we suggest that John does not know why he is being asked to do this activity. As noted earlier, he gives no indication of seeing the required calculation as one involving addition. John is certain that he needs to multiply, but is simply not sure that he has chosen the correct procedure. Whilst he is happy to work on this addition problem with the researcher, he does not indicate that he relates this in any way to the pipefitting task.

For the researcher, and perhaps the reader, with powerful and versatile images for the concept of multiplication, the link between the activity and the mathematical calculation required is an obvious one, yet there is no reason to expect that John will automatically make this connection. John and the researcher are working with two different images for calculating the total take-off length, and as such, even when the correct answer to the addition is achieved, John is not sure what he should now do with this. We do observe John folding back, and recognising that he has an obstacle to overcome. However, he does not link the image making activity he is prompted to engage with to his original difficulty, and he is not able to use his thicker understanding to solve his calculation dilemma. Indeed, we would suggest he is not aware that he may now have a deeper understanding and richer image for multiplication. Although John has folded back, the action is ineffective for him, as he still does not connect repeated addition with the act of performing a multiplication calculation. The researcher continues to probe:

R: *(Pointing to* $3\frac{1}{16}$ *written earlier on sheet)* I'm a little miffed. I don't know how you got that. Show me what numbers you put into your calculator to get that.

J: Ok.

R: I don't know how to use one of these calculators.

J: Ok. Well what I do here, I'm doing a fraction on a calculator. I go, first I go. What's the number I'm doing here.

R: One and three quarters.

J: One and three quarters. I go one, then I hit the fraction button. It gets the little ratio there, right *(referring to the character "r" on the calculator screen)* Three, hit it again. One and three over four. Times, one over three over four. *(As he points to 1r3r4 on the calculator screen)*

R: Ahh!

J: That's a problem? A mistake? Should times it by, times it by two instead of? *(He now enters* $1\frac{3}{4}\times 2$ *into his calculator)* There we go. I did it wrong.

R: Bingo!

J: Got it. Ok. *(He writes* $1\frac{3}{4}\times 2$ *on his sheet)*

R: So you multiplied?

J: Yes.

R: Because you took one and three quarters, and you needed two of them. So instead of multiplying it by two, you multiplied it by one and three quarters. And because one and three quarters is really close to two, you were in the ballpark. But, when you originally wrote down three and one sixteenth you paused, you stopped for a second. And now you've erased it right here. But you were thinking of something and I'm itching to know what was going through your mind there when you were correctly uncertain. You knew.

J: The reason is, the reason is, that I was uncertain, do I multiply it by itself or by two. That's what I was thinking. And I'm thinking maybe, maybe not, maybe.

In this extract John explains to the researcher how he obtained three and one sixteenth, and the researcher becomes aware of the error John has made. John, believing that he has made an error somewhere in his pipe length calculation because it did not match that of another apprentice quickly abandons his original procedure and now multiplies by two, carrying out the correct calculation.

However, we suggest that John still does not connect the procedure for determining the total fitting allowance to one of repeated addition, and how this image relates to the multiplication that he has performed. He knew that either he had to multiply the fraction by itself or by two, but with no understanding of *why* one is correct. For him, it was a choice between the two procedures, and, as he now knows that as multiplying one and three quarters by itself was incorrect, he decides he must instead multiply it by two without offering any explanation as to why this is the case. The researcher probes further:

R: So, *(pointing to crossed out* $3\frac{1}{16}$ *on paper)* did one and three quarters flag that for you, or no matter what you would have got you still would have been thinking about it?

J: I still would have been thinking about it, because I would have known that, I still know in the back of my head, either you times it by, like I'm thinking to myself, times it by itself or you times it by two.

R: Ok.

J: That's what I'm thinking, all the time. So, and I'm looking yesterday's, yesterday's theory, I had no problem doing that. *(Points to written calculations from previous day)*

R: Yeah.

J: I made that in as dummy's terms as I can get. Right. So all I did was change the number here. *(He points to a written computation in notes from the previous day)* The formula still stays the same. And, that's my problem. I didn't want to look at that. I go on memory.

R: Yeah.

J: And what I want to do now is get a fresh sheet of paper and start over again before I cut this pipe. Ok. So that's what I'm doing. I don't want to go any step further, even though I know the answer. That's not going to help me when I do my test.

John is content to now know which operation he should use, and as he comments to "go on memory", although not to simply copy a procedure used previously. His mathematical images for multiplication seem to be unchanged, despite his folding back actions, and we would suggest that although he has now successfully completed the calculation his understanding of *how* and *why* the mathematical operation is the correct interpretation of his visual image is unchanged. Naturally, we question the reliance of an apprentice on procedural memory and return to this in our conclusions. We would suggest that perhaps John would have benefited from a further opportunity to fold back and to initially work with whole numbers, to re-make (or make) an image for multiplication that he could see as being appropriate to the task, and that would help him work with fractional amounts.

Episode 2: Knowing How and Knowing Why – Effective Folding Back

A few minutes later in the same session, with the correct value for the length of pipe now calculated, John went on to measure out the length of pipe that he needed using an imperial units tape measure, prior to cutting it. He is talking with the researcher, and Steve, another apprentice in the group. He has called the researcher over, as he is puzzled about the length of pipe he is measuring.
(John, Steve, Researcher)

J: What we're trying to find out is, this pipe size should be, what was it again?

S: Eight and three eighths.

J: Eight and three eighths.

R: That looks perfect. *(Referring to length of pipe being measured)*

J: So, now, (*pointing to* $8\frac{1}{8}$ *on the tape measure*) that's one eighths, right?

R Yeah

J: Two eighths. (*Pointing to* $\frac{3}{16}$*)* Three eighths. (*Pointing to* $\frac{2}{8}$) Right?

Here John has the correct length of pipe from his calculation, but thinks that he is incorrect when measuring a pipe of this very length. This is because when needing to count in eighths on the tape measure, he actually counts partly in sixteenths (also marked on the tape), thus resulting in him reading an incorrect measurement from the tape. The researcher points this out to him:

R: Yeah. Now wait a sec. There's a problem. Make sure you're not counting sixteenths.
J: OK.
S: Those are sixteenths.
R: OK. This from here to here is a half, right?
J: Right.
R: From here to here is? *(Pointing to the interval on the tape measure bounded by* 8*" and* $8\frac{1}{4}$*")*
J: A quarter.
R: A quarter, yeah. From here to here is? *(Pointing to the interval on the tape measure bounded by 8" and* $8\frac{1}{8}$*")*
J: *(Pause)* That's, the little, the little. *(Pointing to the mark on the tape measure indicating* $8\frac{1}{8}$*")* That's one eighth.
R: Yeah, like if this is a quarter, an eighth is.
J: I know that's a sixteenth. *(Pointing to a small vertical line on the tape measure indicating a sixteenth between eighths)*
R: Half of an eighth. Or, sorry, an eighth is half of a quarter. That's a quarter, then an eighth is from here *(indicating on tape measure)* to here.
S: So every second line is one sixteenth?
J: Right.
R: Every second line. And every single line here is a?
J: Eighth.
S: Thirty second. *(Spoken simultaneously with John)*
R: Every single line?
J: Every line is one is an eighth, is it not? The small line?
S: Every line is a thirty second.

Here we see John and Steve giving different answers to the questions, using their existing images for fractions. However, while both correctly use the language of fractions in this context, it is not clear at any point that John is actually thinking in terms of equal parts of a whole inch, and thus that to measure involves a comparison with the various fractional units that are superimposed upon one another on the tape. Instead, John seems to have and be using an image based on the fact that measuring tapes use different length lines to represent the different fractional divisions of each inch (i.e. very short vertical lines for thirty-seconds, becoming progressively longer towards the half inch), which does not encapsulate an understanding based on a part-whole fractional relationship.

John's confusion is apparent at the end of the extract when he incorrectly states that every single line is equivalent to one eighth, whereas Steve seems more comfortable with the fractional scale and gives the correct answer of one thirty-second. At no time does John talk about an inch being divided into a given number of equal sized parts, which is an image that could help him to more easily work with the complex measuring tape. John continues his explanation to the researcher:

J: Each line here, *(pointing to the fine makings on the tape measure)* see how, differences between the small and the little taller ones, right? I'm trying to make it as, here we go. *(He grabs a piece of paper and holds it over the tape measure so that only the end points of the marking lines are visible above the paper)* Here we go, ok. See how every line is different here, right?
R: Mmhu.
J: Cause one line that's smaller than the others. So.

R: Yeah.
J: That's one sixteenth, a big one is eighth, correct?

Here again John explains the different fractional units by referring to the size of the line used to make the sub-divisions on the tape measure. Sensing that this might be problematic, and that it is John's limited image for fractional units that might be causing the difficulty, the researcher introduces a set of ruler scales printed on acetate that can be stacked to illustrate how fractions are represented on a measuring tape (See figure 3).

Each acetate rule layer has the inches divided up into a different fractional unit, so the first rule has only inches on it, the next shows half inches, the next has quarter inches and so on. This provides a visual image for how a standard tape measure actually incorporates a number of different fractional units superimposed onto one scale.

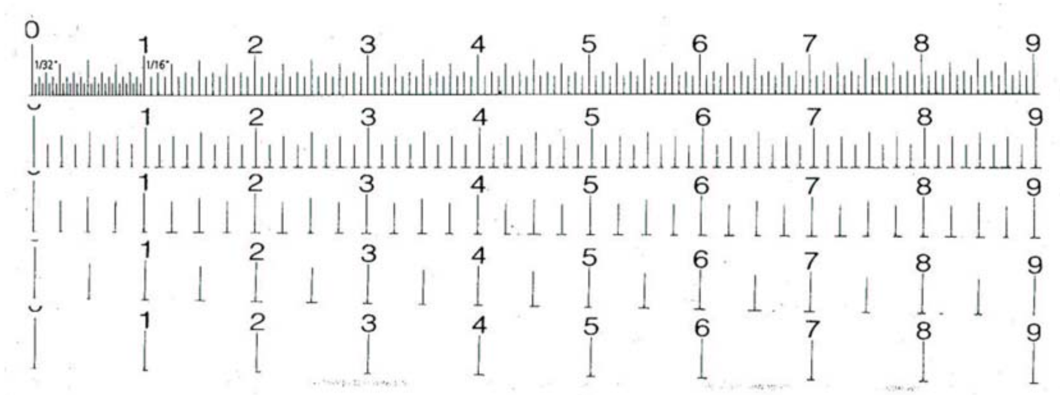

Figure 3. The set of acetate rulers.

The researcher offers the rulers to John, and they start to work with them.

R: Ok. So eight and a half is there. Halves are this ruler here. *(He selects the acetate with only half inch intervals indicated)* Then, I just want to, ok, ok. Line those up. *(The one inch interval acetate and the half inch interval acetate)* I should actually tell you, it's not quite, quite perfect. *(Referring to the acetates teaching tool)* It's out by a thirty second over nine inches, but. Ok. The quarter of an inch is there and it lays overtop like that. Yeah? Eighth of an inch, just look at the eighth of an inch *(indicating on the acetate ruler)* there. *(He positions the eighth inch interval ruler on top of the whole inch, half inch and quarter inch acetate rulers already stacked, one on top of another, on the workbench)* See how, how you're counting?
J: Right, ok.
R: That lays on top like that. But you only need eighth of an inch, right?
J: Right, I need eighth of an inch.
R: So, lets look at your ruler *(the tape measure)* up against this. *(The stack of acetates-teaching tool)* So I'm going to line up the eighth of a, lay it down flat.
J: OK.
R: Look at where your eighth?
J: So that's one, two, three. *(Counting off lines at* $8\frac{1}{8}$*,* $8\frac{3}{8}$ *and* $8\frac{5}{8}$ *on the acetate ruler, shown in figure 4)* Correct? Am I counting that correct?

Figure 4. John indicating what he thinks is $\frac{1}{8}$, $\frac{2}{8}$ and $\frac{3}{8}$ of an inch.

The set of acetate rulers are used here to offer John a valuable image making tool, allowing him to physically manipulate a set of representations of the different fractional units and to see how these relate, both to each other and to the standard measuring tape. Using the resource also exposes the problem that John is having in working with these fractional units. When he comes to count three-eighths, he points to one-eighth correctly, but then continues to point to the lines of the same length as this, i.e. three-eighths and five-eighths. He does not count two-eighths or four-eighths, as these are equivalent to one quarter and to one half respectively, and thus on the ruler are represented by lines of different lengths.

We see clearly here that John did not have an image for a fraction as a location on his measuring tape, as a point on a number line, and he does not see the relationship of the numerator and denominator of a written fraction to the part-whole of an actual inch. As suggested, John's image here for units of measurement seems to be that the lengths of the lines on the ruler scale correspond directly to particular fractional units, and is thus based on the labelling of points on the scale rather than on a continuous scale. The researcher responds:

R: No, you're doing spaces. Count spaces.
J: Ok. So that's one.
R: One, Yeah.
J: Two. *(Now counting eighth inch intervals on the acetate eighths ruler)*
R: Two, and at the end of that space is your?
J: Is this right here? *(He indicates* $8\frac{3}{8}$ *point on his tape measure)*
R: Right there.
J: Ok. Ok. So. *(He re- enacts the stepping process with his pencil tip from 8 inches on his tape measure)* Ok. Didn't see it, here we go. I got you. *(Now explains it back to the researcher)* Ok, so, this is our one, our one-sixteenth right here, correct? *(Pointing to* $8\frac{1}{8}$ *point on tape measure)* No, our one eighth, am I out?
R: Which is it?
J: This one right here, the big one? *(Pointing to* $8\frac{1}{8}$ *on the tape measure)*
R: Which is it, eighth or sixteenth?
J: That's eighth.
R: Ok.
J: Ok. This one here is sixteenth. *(Pointing to* $8\frac{1}{16}$ *point on tape measure)*
R: Yeah.
J: This one here is thirty-two? (*Indicating* $8\frac{1}{32}$ *on the tape measure*)
R: Thirty-second. Yeah.

J: Thirty-second. Am I on the ball with that?
R: Absolutely.
J: Ok.

In this extract the researcher engages John in a directed image making activity, he tells him to count spaces, and John carries out this physical action. In doing this, we suggest that John has now folded back to do something physical that will allow him to modify his inappropriate image for the fractional scale. He returns to Image Making, and is trying to make sense of how the ruler is to be correctly read, and *why*. After counting with his pen on the ruler and reaching eight inches and three-eighths, he re-enacts this stepping process, suggesting that he is now actively making an appropriate image, and is able to count-on in fractions whilst making the correct one-to-one correspondence with the ruler scale. The physical act of working with a manipulative allows him to understand what he is doing, and towards the end of the extract it seems that he now has an image for a fraction as a point on a continuous number line, as printed on the ruler. He is able to confidently identify the other fractional units, though we do not see him count on in these. We contend that John now has a more appropriate and useful image for the concept, though not necessarily complete, that will allow his understanding to grow, and enable him to correctly complete measurement tasks. The researcher probes a little more:

R: Because, that's how many of those little spaces would fit in a whole inch.
J: Right. So if we went one.
R: What are we counting now? What kind of fraction?
J: Eight, eight. Right?
R: Ok. Eighths.
J: So that's one-eighth from here *(indicating the 8 inch point)* to here. *(Indicating* $8\frac{1}{8}$ *inch point on tape measure)* That's two eighths *(pointing to the interval between* $8\frac{1}{8}$ *inch to* $8\frac{1}{4}$ *inch)* and that's three eighths. *(Pointing to the interval between* $8\frac{1}{4}$ *inch to* $8\frac{3}{8}$ *inch)* Gotyah. Ok. I understand now. Now that I can actually express it and point it out, ok.

Again in this final extract we see John confidently counting on in eighths, and recognising that he has an understanding of what he is doing. It is not totally clear here whether he fully grasps the idea of "how many of those little spaces would fit in a whole inch" that is articulated by the researcher. However John does agree when the researcher offers this explanation, so it does seem to make sense to him. We suggest that the future exploration of different fractional units will help him to develop his possibly partially formed image to be one that he can confidently use and apply whatever he is asked to measure, or indeed to work with fractions in another context.

In this episode John's folding back to Image Making has been effective and through his physical act of working with the measuring tape he now has a useful and appropriate image for imperial units of measure, based on a deeper understanding of the part-whole relationship of fractions. This image allowed him to successfully overcome his difficulty with measuring and enabled him to correctly measure the length of pipe, something that he was not able to do at the start of the episode.

Discussion

It is beyond the scope of this paper to comment in any depth on the complex role that mathematical understandings, images and folding back play in the trades training process, but we suggest that trades educators should expect that their trainees may not come with a useful and easily applied range of images for required mathematical concepts needed in their training.

Observing mathematical actions and activity through the lens of the Pirie-Kieren theory, and particularly through attending to the notion of folding back offers a way to both identify potential moment of breakdown, or disconnection in a pathway of growing understanding, and also to anticipate and potentially to overcome these. Recognising where learner held images are localised or limited, and unlikely to be helpful in the new specific workplace context identifies points where apprentices may need a space to be created for folding back to occur.

It would seem that offering opportunities for apprentices to fold back and to engage in appropriate image making activities for some mathematical concepts would be an appropriate way to occasion their growth of understanding, and enable the development of more widely applicable skills. However, our discussion of John also highlights the importance of ensuring that any such folding back act is effective and that the learner is able to recognise not only the limited nature of their existing understandings but also how any inner layer action is relevant to the overcoming of the initial difficulty. Whilst we recognise that in some ways, returning to "play around" with physical manipulatives might seem to be both a backward step and time consuming, we do believe that there is a need to re-engage with some basic mathematical concepts, but within the new context of the workplace. As Forman and Steen (1995) noted, there is a need in the workplace for "concrete mathematics, built on advanced applications of elementary mathematics rather than on elementary applications of advanced mathematics (p. 228). Certainly for John, he was being asked to use relatively elementary mathematical concepts but to use these in problem solving contexts that are very different from those in which the concepts will have been taught or used in school, and for which deep and flexible understandings and images are likely to be required.

It is not clear whether John had some existing images for the mathematical concepts required to successfully complete the pipefitting task (even though we can assume he will have met these concepts in school), but it is clear that even if he did, he did not see these as appropriate or useful in the creation of an appropriate mathematical model for the problem. We would suggest that in the first episode John would have benefited from an opportunity to fold back and to work with whole numbers, to re-make (or make) an image for multiplication as repeated addition that he could see as being appropriate to the task, and that would help him work with fractional amounts. We do see John folding back to image making in the second episode, and through working with the set of acetate rules and having an opportunity to play around with the different fractional units he does seem to have an appropriate image for imperial units at the end of the session. This kind of activity and accompanying learning tools are invaluable for encouraging mathematical understanding that goes beyond being able to merely read a scale or operate on numbers.

Clearly, the measuring tape is a fundamental part of working in pipe trades, and the ability to use this, and to understand the mathematics that is captured by this tool is essential for a worker. Whilst we acknowledge that such understandings are not likely to be made explicit during every task, the possession of a powerful and flexible set of mathematical images related to this offers something to fold back to, should memory fail, or the need arise to work in a new application. Such situations are examples of the kind of "breakdown" identified by Pozzi, Noss, and Hoyles (1998), and are cases where, quite suddenly, the mathematics captured within the tools of the workplace needs to be made explicit. Folding back offers a means through which the encountering of such a need does not necessarily need to result in a breakdown in the growth of understanding, but instead can act as a stimulus for continued growth and progress. Certainly for John, being able to connect multiplication with the image of placing the two cut-off pieces of pipe together, and of then understanding how this can be represented on a measuring tape could have been a valuable and growth enabling experience, in a similar way that was seen to occur when working with the acetate rules in the second episode.

We would also note that it is unlikely that a single act of folding back will ever lead to an image that can immediately be recognised as complete in some way. In our view of mathematical understanding, we suggest that images are made over time and across contexts, and as such should be constantly evolving to meet the new demands being placed upon them. As discussed, we do see John starting to make and have an appropriate image for the task he is engaged with, but for this image to become more flexible and general, and one that he can easily and comfortably use, repeated folding back is likely to be necessary. We hope that this would occur through being asked to work with different fractional units of imperial measurement in a range of different tasks and contexts.

Although we have not discussed them in this paper, the layers of the Pirie-Kieren beyond Image Having are those of Property Noticing and Formalising. At these layers, a learner is likely to have an understanding of a concept that is more general and abstract, and as such applicable to a wider range of contexts. (Working at these layers involves recognising connections across one's images, seeing general properties of these, and then being able to state these in more formal terms – perhaps as an algebraic rule, but with an understanding of why such statements are true). We do not see John yet having such an understanding for fractions, though through folding back he is working towards this more powerful way of thinking.

We contend that in the technical training classroom there is a need, through folding back, to re-visit concepts such as addition, multiplication and fractions and to go beyond learning merely how to operate on and with numbers. As Wedege (2000a) noted there are three different levels at which to consider mathematics in the workplace; the level of skills; the level of understanding; and the level of identity. Our research suggests that apprenticeship training focuses primarily, and successfully, on the first of these levels, whereas there is also a need to develop more general mathematics knowledge that is not so tightly tied to a specific workplace task.

The importance of developing mathematical understanding during workplace training is echoed by Straesser (2000b) who writes "if society is interested in highly qualified, self-reliant workers it may be worthwhile to continue training and education beyond the narrow necessities of the actual workplace" (p. 244) and more specifically that "if mathematics is taught as a bridge between the concrete, may-be vocational situation and the abstract may-be systematic structure, even classroom vocational education can show mathematics as a 'general' tool which is larger an importance that just coping with the narrow tasks of the everyday work practice or the inculcation of algorithms" (Straesser, 2000a, pp. 72-73). Our study supports this view and finds that in particular, there is a need to explore the existing understandings that trainees bring with them, to consider the appropriateness of these images for vocational related tasks, and through the provision of opportunities for effective folding back enable the construction of new images as needed, drawing upon the use of common workplace tools and resources as appropriate, whilst also ensuring that such images are flexible and adaptable.

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[2] For more detail on the different forms of folding back see Martin, Pirie & Kieren, 1996; Martin, 1999; Pirie & Martin, 2000.

Mathematical Autobiography Among College Learners in the United States

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Abstract

This study examines the K-12 mathematical experiences of U.S. university students via an expressive writing assignment: a mathematical autobiography essay. The essays of 67 college students, out of over 300 enrolled in 16 sections of a college liberal arts mathematics course, were analyzed using constant-comparative methods. Two categories of experience connected to aspects of mathematical self-regulation emerged as significant: (1) locus of control for mathematics knowledge and learning; (2) self-evaluations of mathematical ability, efficacy, and potential. Interviews of 18 of the 67 students provided support and clarification for the analysis. An argument grounded in existing research for increased mathematical self-regulation as a result of completing the mathematics autobiography is made. Finally, connections are drawn between learning and psychological theories to support the assertion that using the assignment may help build pedagogical content knowledge among novice college mathematics teachers.

Introduction

Given the constructivist pedagogical orientation of curricular change in U.S. schools over the last 25 years (National Council of Teachers of Mathematics [NCTM], 1980, 1989, 2000), a shift from teacher-regulation to student self-regulation in mathematical learning has been inevitable. Development of the kinds of “mathematical habits of mind” called for by constructivist pedagogies must be folded into school and college instruction (De Corte, Greer, & Verschaffel, 1996; Schoenfeld, 1992). How can this be done? Answering this question requires an understanding of reflection, intentionality, and habits in mathematical contexts and gives rise to many more questions: What is the nature of the relationship among reflective, intentional, and habitual facets of the intended curriculum (NCTM, 1989, 2000) and of the perceived curriculum (i.e., what is reported by students who have experienced it)? How will these experiences inform the next generation of teachers, parents, and curriculum developers in the United States? Even more fundamentally, just what *is* the nature of the lived experience of mathematical learning for this next generation?

The work presented here is a phenomenological study of the mathematical memories of people who were in middle and high school during the 1990’s, when the NCTM recommendations were being implemented. In addition to giving voice to young adults in the U.S., the study examines one method for fostering reflective awareness of mathematical habits among college students: mathematical autobiography. In particular, the study addresses the questions:

1. What experiences and perceptions of reflection, intention, and habit in learning mathematics do young adults in the U.S. bring with them from school to their college mathematics service

courses (e.g., mathematics for prospective elementary school teachers, mathematics for liberal arts)?
2. Does the use of the mathematical autobiography as an expressive writing tool activate reflection in any useful way?

Below, an overview of the theories from psychology, cognitive science, and memory research informing the study is succeeded by a précis of methods. Supported by evidence drawn from essays and interviews, the results and discussion address reflection, intention, and habit through contentions about students' locus of control for mathematical sense-making and their self-evaluations of ability, efficacy, and potential in mathematics. In closing, implications for theory development and K-16 teaching practice are discussed.

Theoretical Framework

Research at all grade levels has suggested that mathematics learning requires sufficient available working memory, pertinent domain-specific skills, and the ability to use them flexibly (Bandura 1997; Boaler, 1999; Boekaerts, Pintrich, & Zeidner, 2000; Darke, 1988; Selden, Selden, Hauk, & Mason, 2000; Malmivuori, 2001; Pajares & Schunk, 2002;). Given a problem situation, self-evaluations of one's ability to generate a solution are activated. Simultaneously, affective self-conceptions are aroused. A learner's assessment of how to balance the resulting mental and emotional demands leads to a self-regulating decision that may or may not be consciously considered: control emotion and put effort into cognition (learning intention) or limit cognition and put effort into preventing "distortions of well-being" (coping intention) (Boekaerts, 1995, 1997). This theorized dichotomy of intention is consistent with brain-imaging research on reasoning tasks showing different parts of the brain active when different intentions are realized (Goel & Dolan, 2003).

Social cognitive theory asserts that human achievement is part of a complex self-system of interconnected Personal, Behavioral, and Environmental factors through which people filter experiences, build understanding, make decisions, and act (see Figure 1; Bandura, 1997). In the context of mathematics learning, this self-system is built up from new and past mathematical experiences (Malmivuori, 2000).

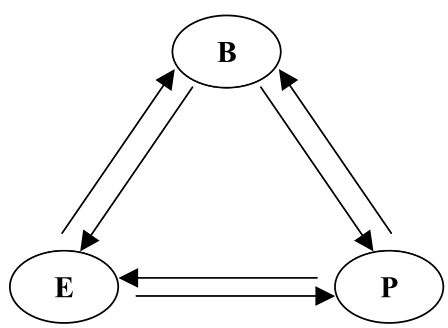

Figure 1. Social cognitive model for triadic reciprocal causation: **B** is for behavior; **P** represents personal factors; **E** is environmental factors (Bandura, 1997).

Emotions, cognitions, meta-cognition, meta-affect, and physiological states as well as perceptions of Environments and Behaviors are integrated at the Personal node into the cycle of attending, organizing, rehearsing, goal-setting, self-evaluating, and acting intentionally that constitutes *self-regulation* (Boekaerts et al., 2000; Schunk & Zimmerman, 1994). An important reflective component of self-regulation is *self-efficacy*: a constellation of one's perceptions about "what one can do under different sets of conditions with whatever skills one possesses"

(Bandura, 1997). Self-efficacy is not a measure of skill, it is perception about task-specific competence. To clarify, consider a student confronted with a pop-quiz: “Solve $3x + 7 = 8$ for x.” The student may not be able to bring to mind immediately any strategy other than one used, unsuccessfully, in the past. Awareness of her own competencies in solving such equations means she recognizes a likelihood of task-failure. Additional levels of affective arousal like anger at having an unexpected quiz, fear of the consequences of failing, or emotions from memories elicited by the activity may accompany her self-efficacy judgment.

If conceptual structures like concept images (Tall & Vinner, 1981) or schema (Dubinsky, 2000) are sparse, then significant effort may be required to understand a problem-statement before considering goal-setting regarding its solution. A student may use up a great deal of working memory deciding *which* knowledge to use (Mason & Spence, 1999). The dissonance between cognitive and affective demands may overwhelm her into the coping state of inaction that is a hallmark of mathematics anxiety (Hall, 2002; Robertson, 1991; Tobias, 1993). Effective self-regulation on the student’s part would include managing her responses to focus on applying an existing strategy or attempting something new in a purposeful way.

More generalized than self-efficacy, *self-concept of ability* may depend on comparison to the abilities of others and include variable feelings of self-worth associated with performance (Seegars & Boekaerts, 1996). In the pop-quiz example above, the student might glance around the room and gauge the apparent ease of working the problem among her peers. If everyone has finished, her conception of her ability may suffer (by comparison). On the other hand, if she sees those around her struggling to complete the problem, though her efficacy self-evaluation is unlikely to change, a positive sense of relative ability could help in regulating volitional response and might lead to her persisting in the problem situation.

The degree to which one identifies with a culture affects the purposes, structure, distribution, and manifestation of self-evaluations. People from highly individualistic cultures perceive themselves to be most efficacious when working alone while those from more collectivist or consensus-based cultures feel they work most productively in group settings that value mutual pursuit or like-mindedness (Fullilove & Treisman, 1990; Oyserman, Coon, & Kemmelmeier, 2002; Zaccaro, 1987). Therefore, it is important to note that depending on personal and classroom culture, self-efficacy, self-concept, and self-regulation can interact in different ways.

Regardless of cultural underpinnings, the impacts of efficacy and ability self-perceptions on cognitive efforts may be most volatile when something is first being learned (Trawick & Corno, 1995; Winne, 1995). Students just building skills and knowledge must attend to, coordinate, and regulate multiple thoughts, feelings, motivations, and intentions simultaneously. Over time, as knowledge is structured, tasks that become familiar may require less exertion. A shift in the perceived effort needed for success may allow students to feel “ownership” of their achievement, a key component identifying an “internal” *locus of control* (Pajares & Schunk, 2002; Szydlik, 2000).

Autobiographical memory and expressive writing

Research on autobiographical memory contends that remembrances of past events, behaviors, thoughts, and feelings is largely determined by current self-image and views about its stability (Conway, 1990; Kotre, 1995). So, if a learner believes himself to always have been pretty much how he is now, the tenor and content of recalled memories is likely to correlate with his current views. Certain conditions can influence such recall bias. In particular, recall was more accurate among U.S. college student populations when a reward for accuracy was offered (e.g., a grade) and susceptibility to current-self bias reduced when a deliberate search of memory was prompted (Cantor, Pittman, & Jones, 1982; Conway, 1990; Rubin, 1996). The results of student interviews and journaling reported in some studies (Borasi & Rose, 1989; Carlson, 1999; Davis, 1997; Millsaps, 2000; Nimier, 1993) may be more representative of current-self view than past-self re-view. Extended reflective or expressive writing is likely to yield distinctions between current-self and recollective reporting (e.g., Brandau, 1988). Consequently, the mathematical autobiography assignment used for this report included prompts for reflective memory searching and was 10% of the course grade.

One major theory of autobiographical memory proposes that the self is actually a collection of “possible selves” including who one has been, is now, and may potentially be in the future (Markus & Nurius, 1986). Additionally, the cognitive-affective duality of autobiographical memory suggests that emotionally laden (positive or negative) memories significant in defining past selves can contribute to current- and future-self views and that these memories can be kindled by cognitive engagement (Folkman & Moskowitz, 2000; King, 2002).

Some research suggests that organizing and reporting memories, especially negative ones, may help in dealing with stress (Cameron & Nicholls, 1998; Pennebaker, 1993). Construction of a coherent narrative from poorly organized recollections of past mathematical selves allows for the repackaging of experiential keepsakes into streamlined memory structures that may be regulated more efficiently (Conway & Pleydell-Pearce, 2000). Working memory that had been used to deal with intrusive cognitions and emotions sparked by interaction with mathematics may be freed for use in current cognitive demands. However, the effect may not be immediate nor rely solely on recounting negative experiences (King, 2002). Klein (2002) reported that though improvements in working memory were small one week after an expressive writing assignment, they were statistically significant seven weeks after the assignment: the greatest increase in working memory (11%) was among college students who had written about a negative experience, a smaller increase (4%) occurred for students who had written about a positive experience. No significant change in working memory was evidenced among students who had written about time-management instead of events of personal significance.

What students offer in their mathematical autobiographies are real memories; whether or not they are precise and fully accurate memories of real events may be debatable (Weingardt, Loftus, & Lindsay, 1995). However, it may not matter which they are. Memories of both types shape the way a person perceives experience, conceives of the world, regulates cognitive and emotional responses, and interacts with others.

Method

The study was conducted at a private comprehensive university of approximately 2500 undergraduate and 700 graduate students in the western United States. The content of the one semester liberal arts mathematics course in which participants were students is summarized in Appendix A. The assignment of the mathematical autobiography and grading rubric was made by way of a web page (Appendix B). Assigned in the first month of each 14-week school term, essays were collected, graded, and returned within two weeks. Throughout the course students were encouraged by verbal and written means to be reflective and self-evaluative when engaging with mathematics.

Researcher

At the time of the study (1995-1999) the author was an assistant professor of mathematics. Her background included five years as a secondary school teacher of English and mathematics, a Ph.D. in mathematics, and a post-doctoral fellowship in mathematics education. The author’s own mathematical autobiography was filled with stories of great sadness, joy, pain, and loneliness in relation to mathematics: from the first grade where her punishment for talking too much in class was to be slapped and ordered to teach Ruben how to add, to failing high school geometry, to subsequently completing a year of geometry in six weeks of self-study, to teaching high school, to being the only woman in a sea of men at her Ph.D. graduation ceremony.

The author’s own efforts to exercise self-regulation, in the face of years of dislike for the syntax and culture of mathematics, was developed enough to result in a Ph.D. in mathematics from a Research 1 institution.[1] The author was the product of an upbringing steeped in diversity and low- to mid-socio-economic status. This background for the researcher is offered to allow the reader a glimpse of the affinity the author acknowledges she felt with her math-avoiding and math-indifferent undergraduate students. On the other hand, the author's enculturation was distinct from that of the majority of her students; most were reared in middle-class, suburban, European-American surroundings.

Textual Data Collection

In total, 324 mathematical autobiography essays were produced from Fall 1995 through Spring 1999 by students in the 16 sections of liberal arts mathematics taught by the author; 318 students gave permission for their work to be used anonymously in research. Of these, 293 essays met the minimum requirements of the original assignment.[2] For the study, a sample of 67 papers was chosen from the pool of 293. First, certain student characteristics were identified for each paper: major field of study, gender, socio-ethnic group, age, and course grade. Papers were then separated into four piles, essays by: women who were prospective teachers (110 papers), women who were not (87), men who were prospective teachers (22), and men who were not (74). Approximately one-third of each pile was chosen randomly (36 of 110 by women prospective teachers, 24 of 87 by non-prospective teacher women; 7 of 22 papers by men who were prospective teachers, 23 of 74 by non-prospective teacher men). Subsequently, papers were selected from these four smaller piles until the distribution of socio-ethnic group, age, and course grade in the sample was as close as possible to that found, collectively, in all papers. The result was a sample of 67 papers.

Participants

A summary of the distribution of the 67 students in the sample by gender and declared field of study is shown in Table 1.[3] Many (45%) planned on entering a primary school teacher credentialing program.[4] The rest of the students were in the humanities (26%), professional programs (15%), or undeclared (10%), with one student studying for a degree in mathematics and one studying computer science.

Table 1. Distribution of student's degree programs in the sample by sex.

	Women (45)	Men (22)	Total (67)
Prospective School Teachers (PST)	56% (25)	23% (5)	45% (30)
Liberal Studies (non PST)	4% (2)	9% (2)	6% (4)
Theater, Art, Music, Dance	11% (5)	9% (2)	10% (7)
Psychology, Sociology, Humanities	9% (4)	14% (3)	10% (7)
Mathematics, Computer Science	2% (1)	4% (1)	3% (2)
Professional, Advertising, Film&TV	9% (4)	27% (6)	15% (10)
No declared field of study (undeclared)	9% (4)	14% (3)	10% (7)

The 67 students had identified themselves in university records as 1% Asian-American, 1% Native-American, 3% Pacific Islander, 4% African-American, 6% Mexican-American, 85% European-American (3% identified with Eastern Europe, 7% with Southern Europe and the remaining 75% with Northern Europe). According to university records, 90% were from suburban or rural, middle-income, homes. The socio-ethnic demographics in the sample were approximately those at the university.

Twice as many women as men were enrolled in the author's courses, a ratio of women to men of approximately 6:3, while the ratio for the other (male) instructors hovered around 6:5. The distribution of students at the institution was 6:4. Several explanations could be posited for this persistent imbalance, the most obvious being that women were more likely to enroll in sections taught by a woman. The ratio 6:3 was present in the sample, so the results reported here are for a population with a higher proportion of women than might be found at otherwise comparable institutions.

Finally, after some discussion with colleagues and students, a student was identified to be non-traditional if he or she had a break of five or more years in full-time enrollment. By this measure, 24% (11 of 45) of the women and 14% (3 of 22) of the men in the sample were "non-traditional" students.

Interview Data Collection

Eighteen of the 67 students were interviewed after their respective courses ended (12 women, 6 men). Originally, 22 students were chosen for interviews based on the preliminary analysis of papers. Of these, 16 agreed to interviews. Two more students were interviewed because they asked to be included in the study when they heard from friends about it. Six interviews were within one semester of the student completing the course, six were within one year, four were in the year following enrollment, two were two years after course completion.

The interviews, conducted over tea and cookies in the author's office, lasted from 40 to 70 minutes for each student. Though interviews were not audio- or video-taped, detailed field notes were taken and immediately typed post-interview. Students were asked for memories about the course. Then, each interviewee was given a copy of her or his essay and asked to read it, comment aloud on it, and respond to detail-oriented, exploratory, and clarifying probes. For example, students were asked to clarify meaning as to whether the sense of "earned," "received," or something else was intended when using "got" in a statement like "I got an A in that class."

Data Analysis

Analysis was through the constant-comparative method commonly used for qualitative data interpretation and theory building (Miles & Huberman, 1994; Strauss & Corbin, 1998). All 293 autobiographies were used in determining 36 thematic categories. The 67 papers in the sample were further analyzed by the author and a research assistant to identify the aspects (dimensions) of categories, and associated sub-categories.

Thematic coding on the pool of 293 essays was completed by the author alone. For triangulation, random samples of 50 and 20 papers, respectively, were thematically coded by two colleagues (with the author's codes in hand). Several terminology differences were resolved by consensus and seven categories revised. Thematic coding was followed by a comparative review of data and categories for the sample. Fellow and hierarchical relationships were investigated via the creation and linking of sub-categories. A sub-category is more specific than a category, giving detail about the who, what, when, where, how, or why of an issue. Through the researcher's comparative interpretations, the relationships in the original collection of 36 thematic categories were integrated and refined to the main results. Preliminary drafts of the manuscript were read and commented on by five undergraduates who had participated in the study. As a result of this member-checking, some changes in the number and length of supporting quotes were made. Unless otherwise indicated, all student quotes (in block quote paragraphs or within quotation marks) are taken, verbatim, from written mathematical autobiographies, indicated by a subscript number *[#],* or interviews *[# - I]*, with *all person and place names fictionalized*.

Results

In their essays, students were very clear about whether or not they considered reported experiences to have been hindrances to their interest and/or pleasure in mathematics (negative) or not. The non-negative experiences included those related as being "positive", "uplifting" or "inspirational" as well as "neutral" experiences that the student perceived as neither furthering interest or liking of mathematics nor hindering it. Analysis of experiences reported included consideration of the context within an essay in which the report occurred (i.e., a playful or sarcastic tone taken by the writer may have led to an assertion being identified as positive when, out of context, it might appear to be negative, and vice versa). Also, checking with students during interviews and manuscript review allowed for verification of contextually-based analysis. Overall, 50% of the experiences described in essays were considered by the student-author to be negative: events and classroom atmospheres which occasioned fear, self-doubt, anxiety, unacceptable (to the student) levels of frustration and, in some instances, physical pain. Student accounts of success (and lack thereof) in mathematics ranged from before birth,

> I was successful doing math in the womb - I divided from one cell into two...that was the last time I was successful in math. *[26]*

to junior high school,

> Locked in my bedroom I would scream at the top of my lungs, `WHO CARES ABOUT THE PROBABILITY OF GRABBING A GREEN MARBLE!!' [18]

to high school,

> Math Analysis...it turned out to be half pre-calculus and half hell! [6]

and college:

> I was 20 and in baby math [pre-algebra]. I didn't talk. I was too afraid of appearing stupid. Therefore, that was exactly what I remained. I didn't speak in a math class until a year and two F's later. [60]

Prospective elementary school teachers accounted for 56% (25/45) of the women and 23% (5/22) of the men. Given their career choice, one might expect the group of prospective teachers to report fewer negative experiences. However, students identified as future teachers were slightly more likely to relate negative associations with mathematics (55% of reported anecdotes). Here is how Karen, a very mathematically able pre-service teacher in her final semester of university put it:

> I always try to find the answer right away and if I can't find it, then I say those three words, "I hate Math." Sure, I'll eventually find the solution, but I'll be frustrated and upset the whole time doing so.[15]

Locus of Control

The notion that mathematical knowledge was something completely external to self pervaded students' writings and interview responses. Students talked about locus of control issues in mathematical sense-making using descriptors like "access," "ownership," "power," and "alien-ness." The perception that the means to accessing and understanding mathematics was controlled by external factors, an *external locus of control*, appeared most in student discussions of three subjects: the nature of evidence in mathematics, cheating, and grades.

Evidence of Authority

The primary property exhibited in student references to mathematical evidence, whether it was convincing and/or persuasive and whether it should be questioned or not, was the preemptive authority of teacher and text. That is, control of the definitions of mathematical ideas and of what counted as evidence was vested in these external sources. Eight students (4 men, 4 women) discussed the idea of coming into some ownership of their understanding and learning of mathematics since coming to college. As Dan, a health sciences student, wrote:

> Up until now [college], I've always had to just accept that what the teacher says is fact but never understood why a formula works the way it does... I've been given the knowledge on what to do, but never what to do with it. In my view, it's like giving a kid a hammer and teaching him how to swing it. But if the kid doesn't know that a hammer is used to drive nails, it's worthless to him. Math has been worthless to me. *[1]*

An awareness of the possibility of shifting away from an utterly external locus of control came through clearest when students discussed geometry – a high school course that for many was a first exposure to mathematics beyond arithmetic. Violet, a prospective elementary school teacher struggling through first-year college mathematics in her final year at university, talked about her high school geometry experience at 15. For Violet, the ultimate arbiters of truth were the teacher and textbook and it was pointless to try to convince an arbiter of truth:

> I had a hard time getting past the fact that the triangles, circles, and squares on the paper were not actually the size that was stated, and why I had to prove something that the teacher already knew was correct. *[28]*

For most of the semester in the liberal arts mathematics course Violet exhibited a fiercely dogmatic disposition regarding mathematics. She was as a collector of mathematical truths and saw her future teacher-role as curator of the collection. Violet reported, as did seven fellow prospective teachers (6 women, 1 man), that the ultimate authority was the "teacher's edition" of the textbook. About six weeks into the course a discussion arose in class around a problem from the text. Students began to discover that the problem was ill-posed (the main reason it had been assigned; Strazkow & Bradshaw, 1995; section 4.2 number 39). In fact, the brief solution in the back of the textbook did not agree with that offered in the student solutions guide. Violet argued vehemently that there was a right answer and that she and her tutor had come upon it. When her classmates refused to accept her assertion, she demanded that the "teacher's edition" of the textbook be consulted. Most students were astounded to learn that the instructor had no "teacher's edition." Several reported feeling lost and betrayed by this lack of authority and a few attempted to project the ultimate authority role onto the instructor, including Violet. When that role was declined firmly, students wailed: "well, then, how do we know what the answer is?!" Eventually they negotiated amongst themselves to the grudging conclusion that there might not be an answer to the question as it was posed. They had to "settle" for relying on themselves for that conclusion.

After the term, the view Violet verbalized in her interview appeared to be a bit more self-aware:

> You have to be on the look out, you know. The people who wrote the book could be wrong, like ... make mistakes. That was new to me, you know. I'd always figured if you don't know, well, go look in the book, in the back of the book. This whole idea of having to think, you know, DURING class ... this was very hard for me. [Humphing noise] Yeah, I still don't like it, I don't know ... I'm starting to not even be sure I want to be a teacher, you know? I mean, all these kids are going to look at me like I'm supposed to know and, well, will I? And if I don't, I'll have go to find out, you know? That's a lot of work! [laughs] *[28-I]*

In addition to demonstrating a growing awareness of the intellectual work involved in being a schoolteacher, this excerpt from Violet's interview illustrates her moving away from the notion of mathematics knowledge as passively received (or, as one student put it, "absorbed") towards a view of it as actively constructed and internally validated. In other student talk about geometry, pre-service teacher Jennifer noted:

> I'm planning on becoming an elementary teacher, but if I were to ever change over to high school, I would like to teach Algebra or geometry (please note, crossing truth tables as a teacher wouldn't be that bad because I will have the teacher's book). *[32]*

Again, the perception that a mathematics teacher is not necessarily one who is knowledgeable, but rather one who has access to external sources of information – as in the teacher's edition of a textbook, arises. Like many pre- and in-service teachers (Spangler, 1992; Tatto, 1999), Jennifer's personal epistemology regarding the nature of mathematics included mathematics content and processes as *all* quite fixed, algorithmic, and "out *there* [external]" *[32]*.

Cheating

Students sometimes identified their contemporaries as more authoritative sources than themselves. An external locus of control for mathematics appeared to include peers as well as teachers for the five (3 women, 2 men) who discussed cheating as a means to gain information on exams. Tanya, a communications student, cheated early and often:

> Third grade was my first time that I got caught cheating. A boy named Griff Peels was really good at math, and I used to sit right by him. A few times, when we were doing work out of [the] textbook and I wasn't understanding a problem I would look at his paper ever so casually. I thought I was being sly, but I got caught. The teacher didn't catch me Griff caught me... At first I denied it, but eventually I told him I had. He told me that he didn't care and that I could when I needed to.*[41]*

The tendency to cheat reasserted itself in her experiences repeatedly until college, at which point she finally failed a course: "I took Math 71 [intermediate algebra] three times before I passed.*[41]*" Interestingly, Tanya's epiphany about the "waste of time and energy involved in cheating*[41]*" happened in her third go of Math 71, in which class the instructor had students share their mathematical histories (aloud, in class). That exercise gave her, she reported, the opportunity to have sufficient information about her Math 71 peers that she could "see I wasn't as dumb as I thought*[41]*" (an adjustment of her self-concept of ability).

Heather, a 20 year-old television journalism student in her second year, started cheating in the second grade, stopped for a while, and picked it back up again in high school pre-algebra. She reported in her interview that the subjects she had cheated in were "math and science, but that's really all the same thing anyway, MATH.*[66-I]*" Her second grade teacher discovered her copying from a fellow student, Jackie, and seated her away from that student for the next exam:

> I remember getting the test back as if it were yesterday. I received no scratch and sniff sticker no gold star just a big "F" in red magic marker. Well there was proof and I was found a cheater in the court of second grade. I then had to spend every recess for two weeks with Jackie Town who I hated more and more by the day listening to her babble on about the rules of subtraction (why she was punished is beyond me). There are two things I learned from this, subtraction, ... and also, if I am going to cheat I might want to change up my answers a little bit from the person I am cheating off of.*[66]*

Heather returned to cheating in high school when she went from public school to a private Catholic school. She remembered the lesson from second grade and reported "My classmates and I had discovered the art of cheating and practiced it ritualistically.*[66]*" She recalled in her interview "successfully cheating my way to a B in Sister Ruth's class....Sister never did catch on yet I realize now that I was only cheating myself. The SAT's proved that.*[66-I]*"

Apparent in these reports of Heather's experiences are indicators of her external locus of control regarding validation of mathematical understanding: her SAT scores "proved" she did not have a sufficient understanding of mathematics. Heather also reported mathematics as something that was "absorbed*[66-I]*" and saw Jackie's tutoring of her as Jackie being "punished." Moreover, she felt betrayed by the episode in Mrs. Forth's class and proud, for a time, of her and her classmates' efficacious use of cheating.

Roberta, a prospective teacher who failed her first exam in 11th grade Algebra, was not proud of her one and only foray into cheating:

> Well, instead of going to my teacher and asking for help, me and my girlfriends decided that I should cheat off of them, so I did. Looking back I have no idea how we were never caught cheating, but we weren't. I remember being nervous and having major anxiety everyday during math class. I also remember that I managed to pass algebra 1 with a B, but I would definitely change this memory if I could.*[9]*

In her interview, Roberta referred to her anxiety about mathematics repeatedly and said she felt it all went back to the fact that algebra was "impossible" for her to understand.

One student discussed cheating for a distinctly different reason. George, a communications student, talked about his sixth grade basic mathematics course where homework was graded on the honor system – the teacher read answers out loud, students scored their own homework and then verbally reported their score when roll was called. His desire to avoid notice by his peers led George to a remarkable, if convoluted, vesting of control with himself, though the locus of control for mathematical success remained with his teacher:

> When I got a tutor I did not want my friends to know because I thought they would think of me as being stupid. All of a sudden my homework grades were much better but when the teacher asked us what we had scored on the homework I would give her a lower grade than I had really received thinking that she would say I was cheating, or not grading my paper correctly.*[12]*

It would appear that some of George's mathematics self-evaluative habits were already fairly well formed by age 12. So, despite knowing he was doing as well as or better than his peers (an increase in self-concept of ability), he felt he could not risk an increase in the perception *others* had of his ability.

Five women and three men reported "copying" the homework of friends. This form of cheating was seen by students as quite different from cheating on a test. On a test, getting an answer from someone else was a way of compensating for a lack of understanding (the result of a self-efficacy judgment of imminent task failure). However, homework copying was seen simply as a way to satisfy the behavioral demands of an external authority (the teacher) since "homework usually didn't have anything to do with really *learning* anything anyway. *[35]*" Though the locus of control for mathematics was still vested in the teacher, the decision to copy another's homework might be seen as a separate, internal, locus of control for grades. The tale told by Marcus, a freshman music student, epitomizes well the statements by those who report homework copying. Marcus moved to a new school in mid-term and had to "catch up":

> Let's be real; what would any high school student do in this position? Copy. Of course. There was no other way for me to succeed. I had all of my homework, flunked the few make up tests, and by the time I caught up and got the hang of things the semester was almost over. The "As" that I earned at the end of the semester balanced out the "Fs" that I earned at the beginning. The result was a "D." So much for being valedictorian. *[18]*

Marcus was one of only two men to use the word "earn" when referring to an F grade; throughout his paper and interview he used the word "earn" consistently and indicated grades were never "received" by anyone. The only other male student to use "earn" in reference to an F grade was seeking a degree in organizational leadership*[11]*. He noted that he had "earned an A" and also said he "received" both an F and a C. In fact, the sub-category of locus of control concerning grades was an especially rich vein to mine in the student papers.

Grades

Students frequently used forms of the word "get" when discussing mathematics grades. After student interview responses had clarified some usages, context analysis was used (cautiously) to determine whether the grade "got" by a student was perceived as *earned* (some attribution of self-responsibility for, or ownership of, learning) or *received* (some attribution of external responsibility for learning). For example, the following excerpt was classified as an *earned* attribution:

> I was having no problem getting all A's in everything except for math. Mrs. Crumble was the meanest, grumpiest teacher that I can ever remember having. I pulled off getting A's for the first three quarters in her class, but I dropped down to a B during the last quarter. I was trying so hard to get an A... I asked her nervously on the last day of class what my grade was and she told me: "You got an A by the skin of your teeth."*[17]*

On the other hand, the following was grouped with the *received* attributions:

> The only way I made it through that class was because I copied all of the answers out of the back of the book and then when tests came I would cram and do alright. The whole year I got A's and a B so everything worked out okay except for that I did not learn one thing. *[67]*

Categorizing student-talk about "received" grades was frequently straightforward:

> My sophomore year I took geometry, which was a very unfortunate experience. I received a B both semesters, but I did not earn them.*[45]*

And sometimes challenging (the following was grouped with "earned" based on the final sentence in the anecdote):

> I was 31 years old but determined to get a college degree. I had a hard time following the program in this class [intermediate algebra at a community college]. I struggled. When I received my first test back I had gotten a D and the girl sitting next to me got an A. I thought, 'what does she have that I don't?' It was a tattoo, a pierced nose, ear, lip, eyebrow, and a mohawk. I dropped that class. I guess I had my hands full with the kids at home, ages 3, 5, 7, and 9. I couldn't concentrate.*[29]*

Sixty-six of the 67 students talked about grades in their essays. Thirty-six (80%) of the female students reported "receiving" grades and 15 (34%) reported "earning" grades (six reported both). Among the male students, 16 of 22 (73%) reported "receiving" grades while 14 (64%) stated they had "earned" grades (nine reported both). One male student (non-pre-service teacher) did not make any reference to grades. Figures 2 and 3 give the frequency of each type of grade in each category by sex.

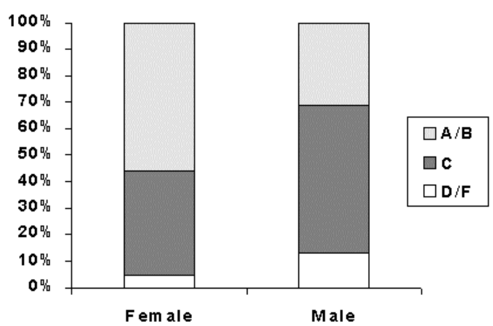

Figure 2. Perceived grades "received."

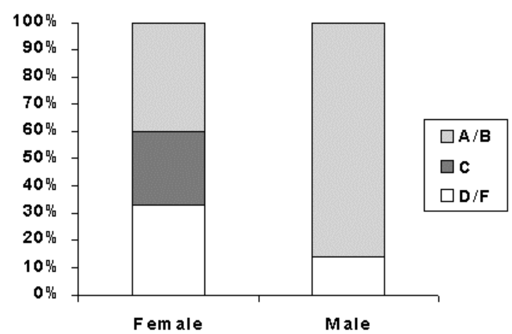

Figure 3. Perceived grades "earned."

Men were much more likely to see themselves as having "earned" high grades while women were more likely to see high grades as "received." Of the 14 men reporting grades in "earned" contexts, *none* reported having "earned" a C. Women who reported grades in "earned" contexts tended to remark most on earning high or low grades. These results are not surprising given similar findings in the research on gender differences regarding perceptions of mathematics achievement among middle and high-school students (Gilbert, 1996; Watt & Eccles, 1999).

Ability, Efficacy, and Potential in Mathematics

In essays and interviews students spoke regularly about emotions, including love, hate, fear, frustration, and fury they felt when mathematically engaged. Jessica, a prospective teacher, echoed 40% of her pre-service teacher peers when she asserted:

> I love it [math] when I succeed and I hate it when I just don't catch the concept. *[63]*

Or, as another female prospective teacher noted:

> … math has been an incredibly hostile force in my life for so long and while I am being dramatic, it has caused me much distress.*[67]*

One interviewee, Amy, a semester after the course in which she wrote her mathematical autobiography commented on an increase in her ability to "manage" mathematics:

> I don't usually like math at all. But, I feel now like I can allow myself to feel that way and that, somehow, makes it so I feel like I can deal with the math when I am doing it. *[16-I]*

Colleen, a capable and self-aware prospective teacher, met every challenge in the liberal arts mathematics course and actively participated in classroom discussions. In her essay she discussed her potential in mathematics:

> I was terrified of Geometry, and thought Calculus was something that I could never achieve, let alone master. Of course once I got in and tackled these subjects, I found that it was not that bad. I am still intimidated by the thought of taking Statistics. ...Well, I have come to learn that I might actually like math, and that I am not half bad at it. *[49]*

Some students also related stories about how their ability self-perceptions influenced career choices. Shelly talked with a career counselor about a budding interest in science early in college:

> I started my first year of college at Sierra Madre Coastal, a wonderful junior college. I took my first science class, Geology, and I loved the course. I have always enjoyed the sciences and found them exciting. I talked to my counsellor about becoming a Geologist and the first horrifying question he asked me was, 'Do you like Math and are you good in the subject of Math?' Shamefully I said no! I wanted to be good in Math but I have had trouble understanding it, I told him. He advised me not to be a Geologist and to stay far away from the sciences because there is too much Math involved. I took my counselor's advice and now I am going to be an elementary school teacher. *[61]*

As disturbing as the final two sentences may sound, Shelly's view of what it took to be an elementary school teacher did shift over the course of a year. Like Violet, she later indicated she had an increased awareness of the intellectual effort required for teaching. Shelly also identified a key point in the decline of her perceived mathematical potential as occurring in the third grade:

> I had not labeled Math a problem until it was pointed out, [by] my third grade teacher, that I had a problem with it. Mrs. Roy, my third grade teacher, destroyed my self perception of Math... [she] told me numerous times that I would never do well in Math, and she was right.*[61]*

In contrast, Layla wrote in her essay that she always thought of herself as doing well in mathematics but a review of old report cards was a revelation to her:

> My teacher, Mrs. Wills was one of my favorite teachers. Her comments to my mom and dad were as follows, "Ordinary Numbers, Layla needs to be much more diligent in this area, April 1981." What are "ordinary numbers", and what does that mean? I must not have been a great participator in math in the first grade. That's what I understand from her comments now. Looking back at this report card now I can't believe I was worse than I thought I was, but it was only the beginning. *[56]*

Layla began her interview with an upbeat, positive view about her relations with mathematics but, after re-reading her essay, declared, "Oh. I guess I did it again, didn't I! Here I am thinking I've always been fine with math but, when I read this, I realize I am not. Very weird."[5] Layla's primary means of dealing with mathematics when she was in the liberal arts mathematics course was to look for rules and, once they were successfully memorized, be satisfied she "had learned." Her approach led to a great many near and total failures on quizzes and exams because she did not have organized structures (e.g., concept or problem situation images; Selden et al., 2000; Tall & Vinner, 1981) to hold and connect all the memorized declarative material. Only after she began being tutored was she able to extend her memorization of "rules like negative and negative is positive" to include process connections: "I have to remember its negative *times* negative that's positive, because for addition it's opposite.*[56-I]*" This expansion was sufficient for her to pass the course with a D. In her interview, Layla referred back to a particular passage of her essay, about high school algebra, repeatedly. She claimed "reliving" the experience was a contributing factor in her decision to change from a career goal of being a teacher to one of being a police officer:

> Mr. Gaspar was my teacher and he was a real jerk. He did nothing to try to help me when I was struggling. My parents had a conference with Mr. Gaspar and he told them he would help me, but every time I went to him for help, someone else was more important than me and he would take a long time to finally answer my questions. It was quite discouraging feeling I couldn't go to my own teacher for help. *[56]*

Layla voiced concern that

> "if [like Mr. Gaspar] I get really overworked, around kids, and get nasty because something [mathematical] doesn't work the way [I think] it's supposed to, it will influence them a whole lot more than if I am dealing with adults...I'd rather deal with the adults. *[56]*"

Jon*[4]*, who was efficacious in mathematics as long as fractions were not involved, transferred from college to college as the meeting of a mathematics requirement "reared its ugly head. *[4]*" He reported that his multiple college transfers were based, in part, on the view of his mathematical potential established by him in fourth grade:

> ... the smartest group was called The Dolphins and there were only a handful of kids in this group. Then there were The Sharks, which comprised most of the class. Then there were The Whales, the slower kids. Then there was a kid from Arkansas, a kid who liked to start fires, and me. We were Plankton. I'm kidding, we were called The Squids, really, in fact I think I named the group myself.*[4]*

Jon wrote that his experiences in mathematics "went downhill from there" and his "utter lack of potential in math" led to choosing a degree in theater. He "changed colleges twice and then avoided the required math" until his final semester at university.

Carlson (1999) reports on successful mathematics graduate students who credit a mentor, typically a high school teacher, with challenging, encouraging and assisting them into the pursuit of a mathematics-related career. The group of students in this study evinced a complementary state: 60% (27 women, 13 men) credited a high school teacher with boring, discouraging, or hampering them in mathematics. This led to what many identified as mathematics-avoiding career choices. This group of 40 "math-avoider" students included 12 prospective elementary school teachers. However, no connection appeared to exist between a student's reported judgments of teachers (as "good," "bad," or "indifferent") and that student's identifying her or himself as a "math-avoider." In fact, four women who were prospective teachers said they valued mathematics "in spite of" a bad or indifferent teacher. They wanted to become good teachers of mathematics *because* they had experience with bad or indifferent teachers.

Discussion

Some answers to the first research question about what experiences and perceptions in learning mathematics young adults in the United States bring to their college mathematics were offered above. In summary:

- *Reflection*. Students reported that negative and non-negative experiences influenced their introspective perceptions of their past, present, and future mathematical selves.
- *Intention.* Students bring to their college mathematics courses the perception that intentional engagement with mathematics is externally driven by Environmental factors like grades and the Behavioral expectations of teachers.

- *Habit.* Students have spent years developing the habit of taking the locus of control in mathematics learning to be external, an Environmental rather than a Personal social cognitive factor. Also, student mathematical experiences appear related to an intimately felt collection of Personal factors based on self-evaluations of relative ability, efficacy, and potential that are connected to decision-making patterns (including long-term decisions about career).

The answer to the second question, whether or not the use of the mathematical autobiography as a curricular extension activated useful *self*-reflection, seems to be "Yes, for some students." The expressive writing exercise appeared to foster a valued sense of perspective. This was exemplified in the excerpts from Shelly*[61]*, Colleen*[49]*, Tanya*[41]*, Violet*[28]*, and Dan*[1]*. That the assignment influenced mathematical self-reflection is also supported by the reports of the 18 interviewees. Nine people, seven of the 12 women and two of the six men interviewed, echoed Amy's*[16]* remark that she felt more self-control around mathematics by allowing herself to write about and feel emotions without getting tangled up in them.

From social cognitive research on negative self-referent thought come strategies for intervention:

> The most powerful way of eliminating intrusive ideation is by gaining mastery over the threats and stressors that repeatedly trigger the perturbing trains of thought. ... [to] equip people with the knowledge, skills, and beliefs of personal efficacy to manage the things that disturb them. (Bandura, 1997)

College mathematics instructors are not psychotherapists. Nonetheless, it seems that recalling, reflecting on, and reframing memories that are part of the "perturbing trains of thought" stimulated by the "stressor" of mathematics may enhance self-regulation. Instead of seeking to rid students of their reactions to mathematics, a mathematical autobiography allows students to acknowledge these responses and build reflective awareness of them. The means to influence possible-self and concomitant self-efficacy views lies in self-regulation. Self-regulation may come in many forms: from response to the anticipation of consequences of behavior choices (Boekaerts et al., 2000), to self-aware constructing of goals (Locke & Latham, 1990), to beneficial inner-speech (Meichenbaum, 1984).

However, it must be noted that the strengthening of aspects of self-regulation requires an appropriately supportive classroom milieu. Without in-class prompts and opportunities that encourage personally owned, self-reliant accomplishment in mathematics, the most significantly impacted students may continue to attribute their successes to external rather than personal, internal, sources (Bandura, 1997; Pajares & Schunk, 2002).

Implications for Theory, Practice, and Future Research

The process of reflecting on and writing about mathematical experiences traverses the pathways in the social-cognitive-model (Figure 1) and appears to act at the Personal node as students reconstruct their memories in the context of their past, present, and future selves. What is more, the paths between Personal, Behavioral, and Environmental nodes can be related to learning goals. Take, for instance, Kirshner's (2002) metaphors of teaching for learning as habituation, as enculturation, and as construction. Curriculum centered on helping students learn by habituation encourages interaction of Environmental and Behavioral factors (E⇔B) by repetitive working of similar problems while minimizing Personal factors. Teaching for learning as enculturation relies on the development and interplay between one's own Personal factors and those of others within task Environments (P⇔E). Teaching for learning as construction is driven by a learner's Behaviors and negotiation of meaning and significance with self and others (B⇔P). The three nodes of social cognitive theory can be seen as transfer points in Kirshner's (2002) cross-disciplinary strategy view (see Figure 4).

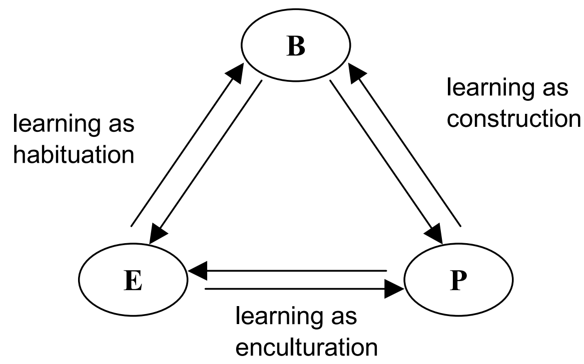

Figure 4. Possible relationship between social cognitive theory (Bandura, 1997) and Kirshner's (2002) crossdisciplinary multiple learning theory.

If, as Kirshner suggests, integrating multiple learning theories leads to "educational efficacy," then understanding how theories may be influenced at the Behavioral, Personal, and Environmental nodes can have practical implications for teaching. If mathematical autobiography acts at the Personal node, it may have more significant impact on college students whose learning has been enculturationist or constructivist in tone since each of these has P as a driving node (see Figure 4).

Benefit to Students

Mathematical autobiography may be especially useful among college populations. Developmentally, the kind of self-reflection and recollective thought called for in the assignment is only deeply possible among late adolescents (Rubin, 1996). Can the assignment harm rather than help? Research on expressive writing at the college level indicates that such an assignment is unlikely to be a hindrance to learning (Hirsch & King, 1983). Among the 300+ students who wrote essays during this study, in no case did the assignment ever appear to have a detrimental effect (though a few did complain about "having to write in a math class!?!").

Benefit to Instructors

Notwithstanding the possible positive outcomes for students, a major benefit arising from the assignment is afforded the instructor. Undergraduates in mathematics service courses are likely to have quite different personal mathematical histories from their instructors. As is the case with new schoolteachers, there is a tendency among novice college instructors (e.g., graduate teaching assistants) to "give up too easily" when difficulties arise in communicating with students (Borko et al., 1992). Part of the development of pedagogical content knowledge for college teachers involves anticipating, and incorporating into teaching, the manifold abilities, experiences, and concerns of students. New college instructors who teach lower division and service courses stand to learn a great deal from reading a few dozen student mathematical autobiographies. For a sufficiently self-aware instructor, such knowledge may improve the clarity of communication with students (Cervone & Peake, 1986).

Future Research

The well-entrenched view that teachers are the active regulators of learning in mathematics and that the student's job is to listen, compute, and get the right answer, fast, have been reported in several places (Sowder, 1998; Spangler, 1992; Thompson, 1992). Researchers have written at length about the cognitive-affective dynamics in school-age learners (DeBellis & Goldin, 1999;

Malmivuori, 2001). However, how the teacher is "active" and student is "passive" view develops and persists has remained relatively unexplored among college populations.

Research currently under way by the author suggests that dissonance between personal and socio-cultural environments in college mathematics classrooms has an impact on the influence(s) of the mathematical autobiography assignment. Further research into the effectiveness of the assignment in relation to a variety of social, cognitive, and affective variables – including the influence of teaching style – is a clear next step.

Finally, it is an open question whether newly acquired self-aware response in college-age learners will be experienced only at the short time scale of local affect (minutes, or at most, weeks), might be a recurring mood or an attitude that persists for a school term, or may be sustained for longer periods. Further investigation, into the interplay of aspects of affect, reflective thought, self-concept of ability, perception of self-efficacy, and achievement among undergraduate students, is needed.

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[1] One classification scheme for universities in the United States identifies institutions based on several criteria, including the level of research conducted by faculty. "Research I" was the designation given to the top research universities in the U.S. (Carnegie Foundation for the Advancement of Teaching, 1994).

[2] An essay was omitted from the pool if it was at least 15% shorter than the minimum length of 1100 words (15 papers). A paper was also omitted if it was not on the assigned topic of mathematical autobiography (8 papers - four were schooldays reminiscence with no reference to mathematics; four were about how arithmetic is used in everyday life). Finally, two papers were omitted due to their marginal comprehensibility (neither student continued at the university).

[3] In Table 1, each percentage is within 10% of the comparable population value. To clarify: 56% of the sample was female prospective teachers; for all sections of the course taught by the author, 61% were female prospective teachers; 56 is within 6.1 (10%) of 61.

[4] In the state where the study was done, teacher credentialing is a post-baccalaureate process. Prospective teachers complete a bachelor's degree and then apply to a teacher-credentialing program. Pre-service teachers typically spend one to two years in methods and pedagogy courses and one semester practice-teaching before becoming credentialed with a teaching certificate.

[5] A reluctance or inability to engage in the self-reflection and recall asked for in the assignment that may be difficult for some and virtually impossible to overcome for a few. Lumley, Tojek, and Macklem (2002) report on people who are repressive or, more extremely, alexithymic (literally, "lack words for feelings").

Appendix A. Content for the Liberal Arts Mathematics course.

The two-semester mathematics requirement for a Liberal Studies degree consisted of one semester of descriptive and inferential statistics in addition to the liberal arts mathematics course. As a result, neither statistics nor probability were included in this course's curriculum.

The textbook for the course, Strazkow and Bradshaw's *The Mathematical Palette* (1995), was augmented by instructor-created materials incorporating the use and programming of calculators (such as Texas Instruments TI-81, 82, 83, 85, Casio 7700G, Sharp 9200/9300, Hewlett-Packard 48G). When the module on infectious diseases was used, that too was through an instructor-constructed packet based on the first chapter of Callahan and Hoffman's *Calculus in Context* (1995).

Major Study Units : All of topics 1-4 and at least two of 5-8.

1. Numbers, Numerals and Words
   * Ancient Systems
   * Hindu-Arabic Systems
   * Basic Number Theory

2. Sets and Logic
   * Sets, Venn Diagram Models
   * Symbolic Logic
   * Inductive and Deductive Reasoning
   * Flowchart Modeling

3. Algebraic and Exponential Models
   * Function notation
   * Linear Models; Linear Programming
   * Quadratic & Polynomial Models
   * Exponential and Logarithmic Models

4. Finance
   * Interest Theory
   * Annuities
   * Loans
   * Present Value

5. Geometry and Art
   * Euclidean and Non-Euclidean Models
   * Perspective
   * Tiling and Tessellation
   * Modeling Nature with Fractals

6. Calculus
   * Functional Difference Quotients
   * Derivatives; Modeling Rate of Change
   * Basic Integration

7. Trigonometry
   * Sine, Cosine, Tangent
   * Modeling with Right Triangles
   * The Laws of Sine and Cosine
   * Modeling with Acute Triangles
   * Circular functions

8. Modeling Infectious Disease
   * Rate Equations; Difference Equations
   * Programming the Equations
   * Predicting Trends; Effect of Quarantine

Appendix B. The assignment web page.

Mathematical Autobiography Project

Please read this entire page!

Preliminary step: Make a list of TWENTY mathematical experiences. For example, what can you recall of learning to count?...of learning to tell time...? of learning what fractions mean?...of learning how to use money? Each person should reach as far back into her/his personal history as possible. Review old report cards; talk to friends, parents, siblings, caretakers, etc. to collect information, anecdotes and experiences. Does your recollection of grades in your mathematics courses match the actual grades on your old report cards? [You might be surprised.]

Draft step: Write a rough draft of at least 850 words (type it, double-spaced) using at least five of the experiences from the list you generated. It is probably best to write it on a computer (and save it to a disk) so that you can edit later and so that you can use the word-count utility most word-processing programs have!

The assignment: Referring to your rough draft and the list generated in the first step, write an essay of 1100 to 3000 words which relates some of the 20 experiences (at least five) in detail. Discuss how those experiences have influenced current attitudes, feelings, thoughts about mathematics and life goals. Include names, locations. For example: "When I was in the ninth grade at Norco High School (that's in Riverside County in Southern California) I had an Algebra teacher named Miss Trimble who sometimes had us do math outside. One incident I recall vividly was the warm, sunny day the whole class went to the football field and we..."

The essay will be graded as follows:

10 points for length: if the paper is less than 1100 words then the length score will be reduced; the scores for grammar and content will be proportionally reduced as well.
15 points for spelling and grammar.
75 points for content: as long as the paper is coherent, is about the student's personal math history and is at least 1100 words long, all content points will be earned.

The instructor is happy to proofread drafts of the paper during office hours.