Useful mathematics for (technical) vocational education

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Introduction.

Based on Freudenthal's ideas about didactics (Freudenthal, 1973, 1991), arithmetic- and mathematics education in the Netherlands is developing in the direction towards Realistic Math Education (RME) since the early seventies. Starting with primary education, it was Freudenthal's wish to reform math education for the whole range from age 4 to age 18. In 1981 the first reform program for secondary education started: the HEWET-project (De Lange, 1987).
In 1997 the first reform program for further vocational education was started: the TWIN-project.
After an impression of the philosophy of RME, a brief description of the TWIN-project will be given with some examples of the content.

Realistic Mathematics education.

In the early seventies Freudenthal introduced a new approach to mathematics education. He claimed that the traditional way of teaching mathematics, in which the start is within the formal system, was anti-didactic (Freudenthal, 1973). Instead of starting at the very end, students should be given the opportunity to re-invent and re-construct mathematical concepts.
In the realistic view, the development of a concept begins with an intuitive exploration by the students, guided by the teacher and the instructional materials, with enough room for the students to develop and use their own informal strategies to attack problems. From there on, the path leads, via structuring-, abstracting- and generalizing activities, to the formalization of the concept.
Contextual problems, both real world problems and realistic problems in the field of mathematics itself, are very important in the RME approach. They serve as starting point for the development of a concept and also as a source for applications and refinements of the mathematical concept.
In RME the students are given enough space to dwell around for some time on a concrete, informal level, when a new mathematical concept is introduced. At this stage in the process students develop and use own strategies for solving problems that are offered within a context. Many times it is observed that, in this informal stage, strategies used by students are different from the ones that mathematicians normally use.
Students try to solve the stated problem and therefore use the given context as a basis for their calculations and reasoning. Most mathematicians will first de-contextualize the problem and then use the formal system to find an answer.
A nice example, taken from lower-secondary education, illustrating this point, is the pocket money problem:

An and Susan are saving money. An already has \$ 50 and receives \$ 2.50 each week, while Susan gets \$ 4 each week and saved \$ 26 up to this moment.

After how many weeks will they have the same amount of money?

Mathematical professionals will immediately translate this little fairy tale into the 'real' world of algebra: $2.5x + 50 = 4x + 26$ with solution $x = 16$

Students who are offered such problems before any theory about linear equations is presented, will choose other strategies.

Create two tables in which the amounts of money are compared week after week. Some will make shortcuts in this table, because regularities are recognized and used.

Or a dynamic setting in which students solve this problem just by reasoning:

At this moment Susan is \$ 24 behind An. Every week she gains back \$ 1.50

So after 24 : 1.50 = 16 weeks they are equal.

Even after linear equations are instructed in a formal, algebraic way, many students will use the context bound strategies to solve contextual problems. For most students the contexts are the real world and the formal system is a not-so-fairy-tale.

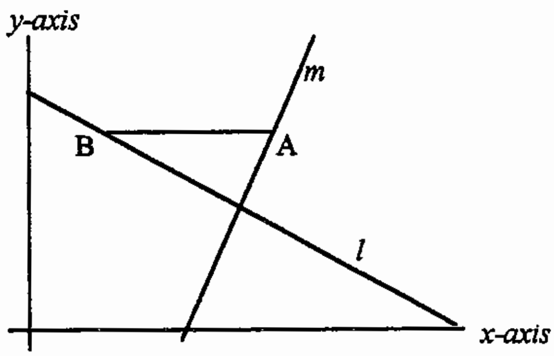
In the algebraic treatment of linear functions in the higher grades of secondary education the same informal, context bound strategies are found again, although the context now is not the real world, but mathematics itself.

The following problem is taken from the mathematics B curriculum (age 17), which prepares higher-secondary students for any exact course in Higher Vocational Studies.

The problem was meant as an exercise for substitution of algebraic expressions, but failed for that purpose. But, much more important, it showed some unexpected student strategies.

Given two lines $l : y = 10 - 0.5x$ and $m : y = 2x - 10$

The horizontal line segment BA connects B on l with A on m and has length 6.



Calculate the coordinates of A and B.

Hint: let $x = x_B$ and express x_A, y_B and y_A in x

The hint was meant to help the students find a way for solving this problem.

Once more we observed that student's ways of thinking about solving strategies are different from the way mathematics professionals are thinking. When the problem is presented to mathematicians without mentioning the hint, most of them do use the approach suggested by the hint.

The students who were offered this problem, did not understand the hint at all and many of them could not solve the problem. But the ones who did by ignoring the hint, showed that they are able to use the basic concepts and skills, learned and developed in an earlier stage, on a higher level.
Four different approaches, each of them little masterpieces, show that the students have become aware of the fact that mathematics is not just a set of isolated subjects like algebra and geometry, and that there are many different ways to attack and solve problems.

Solution 1: *the algebraic way*

From the equations I know that $x_B = 20 - 2y$ and $x_A = 5 + 0.5y$. From $x_A - x_B = 6$, it follows that $5 + 0.5y - (20 - 2y) = 6$, so $y = 8.4$.
Then you easily see $x_B = 3.2$ and $x_A = 9.2$

This approach is closer to the context of the problem than the one suggested by the hint.

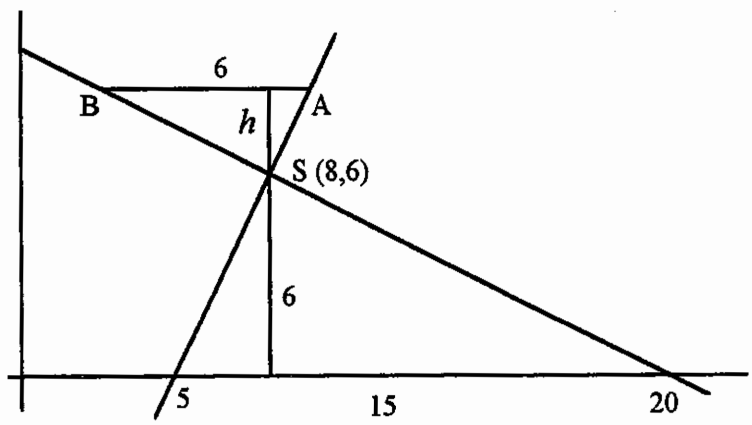
Solution 2: *the dynamic way*

Shift line *m* to the left over six units. This new line *m'* intersects line *l* in B. The formula for *m'* is easily found: a horizontal shift of six units to the left causes a vertical shift of 12 units upwards (because of the slope of line *m*), so line *m'* has equation $y = 2x + 2$. Now intersect the two lines *l* and *m'* to find point *B*, etc.

Two remarks on this dynamic approach:
1. This solution is the dynamic version of the (very static) strategy that was suggested by the hint!
2. For finding the intersection point of the two lines the strategy that was mentioned with the pocket money problem, can be used:
at the *y*-axis the vertical distance between the lines is 20. For every unit going to the right, the distance will decrease by 2.5.
So after going 20 : 2.5 = 8 units to the right, the two lines will meet.

Solution 3: *the geometrical approach*

This problem can be viewed as a geometrical problem, in which similarity leads to the solution

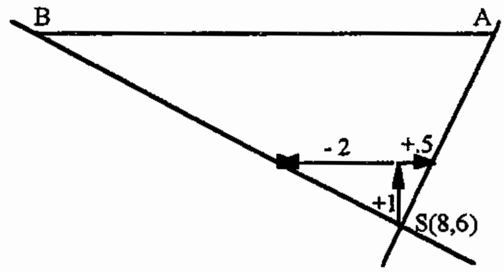


It easily follows that $h : 6 = 6 : 15$, so $h = 2.4$ and $y_A = y_B = 6 + 2.4 = 8.4$ and so on

A solution in which students show that the different strands of mathematics are connected instead of isolated.

Solution 4: *Use the concept of slope*

The slope of line *l* is 2, which means that a horizontal increment of 1 unit causes a vertical increment of 2 units. The other way round: a vertical increment with 1 unit causes a horizontal increment of 0.5. This idea can be used for the stated problem:



Starting in the intersection point S(8,6) a vertical increment of +1 causes a widening of the horizontal gap between the two lines of 2.5 units. The gap has to be 6 (the length of BA), so from S you have to go up for 6 : 2.5 = 2.4 units.

So $y_B = y_A = 6 + 2.4 = 8.4$, $x_B = 8 - 4.8 = 3.2$ and $x_A = 8 + 1.2 = 9.2$

A very bright and high level-understanding of the concept of slope and a flexible use of this concept in a different way.

The two examples, both about the concept of linearity, were chosen because they illustrate some important aspects of RME.

* Learning a mathematical concept by starting with contextual problems not only gives the student the possibility to gain self confidence and really learn to understand the concept, but it can also serve as a basis for further development of that concept. At the same time, the contextual strategy used to solve the problem, is a kind of anchor for solving all kinds of linear equations, including the bare problems given in most math textbooks.

If a teacher really values students' own solution methods, it is possible to improve the quality of both teaching and learning. Using students' strategies to solve problems in a context bound way as a basis for further concept building makes instruction more valuable for (most of) the students.

* Mathematics education that emphasizes an active role for the learners, will definitely lead to some other priorities for wanted learning outcomes.

In the traditional view exercising on an abstract and general level within the formal system is priority number one. In RME the principles of *mathematizing* and *integration of strands* are becoming more and more important.

Students show that a really well funded understanding of a concept makes it a useful tool for new situations (e.g. the dynamic approach and the use of the concept of slope) and that it can also be used in other strands (e.g. recognizing a geometrical problem in an algebraic setting).

The TWIN-project

In 1997 the first cohort of students arrived in vocational courses at the intermediate level, after finishing an RME-based reform program in secondary education. For that reason, but also because mathematics was about to disappear from the vocational

courses, a project was started in 1996 in which a new curriculum is to be developed for technical vocational courses: the TWIN-project. The name TWIN stands for:
Techniek (in English: Technical Vocation)
Wiskunde (Dutch for mathematics)
ICT (Information and Communication Technology)
Natuurkunde (Dutch for Science, more specific: Physics).
Mathematics and science were, like languages, strange topics in vocational education. The students are preparing for any kind of job in the technical sector and these 'general' courses didn't add very much to their practical preparation for work. Consequently, some years ago there was a tendency to put the necessary mathematical techniques in the vocational courses. Only the switch to a more useful math- and science program could save these courses from fading away.
The general goals of the TWIN-project are to create a new curriculum that
- makes sense for the students (not only in math classes) who are preparing for a job
- takes into account the reformed math-program of secondary education
- makes use of new technology for new, alternative ways of learning and doing mathematics
Before starting to write any materials, an inventory was made of important mathematical skills and concepts to support the vocational courses.
Most important finding:
In applications, mathematical models (functions) are merely about proportionality. Starting to discuss functions from proportionality has the advantage that you can use a lot of technical contexts to study relationships between quantities, in which contextual reasoning is possible.
An example.
The power P , supplied by a windmill, is directly proportional to the square of the diameter D. In a formula: $P = a \cdot D^2$
An important question about proportionality is:

>> What happens to the power P when you double (halve) the diameter of a mill?

The supplied power is also directly proportional to the third power of the wind velocity V.
In a formula: $P = b \cdot V^3$.
An relevant question that is not easy to answer with use of mathematical arguments:

>> P is directly proportional to the square of D and also to the third power of V.
Which one of the following formulas is correct:
$P = c \cdot (D^2 + V^3)$ or $P = c \cdot D^2 \cdot V^3$

There are two simple arguments for students to choose the right formula. Both of them are not mathematical arguments.
One is from a physical point of view: you can't add two different quantities in a formula that do have different units of measurement. It's like adding apples and pears.
The other one is a common sense argument within the context of the mills. Suppose there is no wind. The first formula says that the mill still supplies power, without working. So both arguments say that the second one is the right formula.

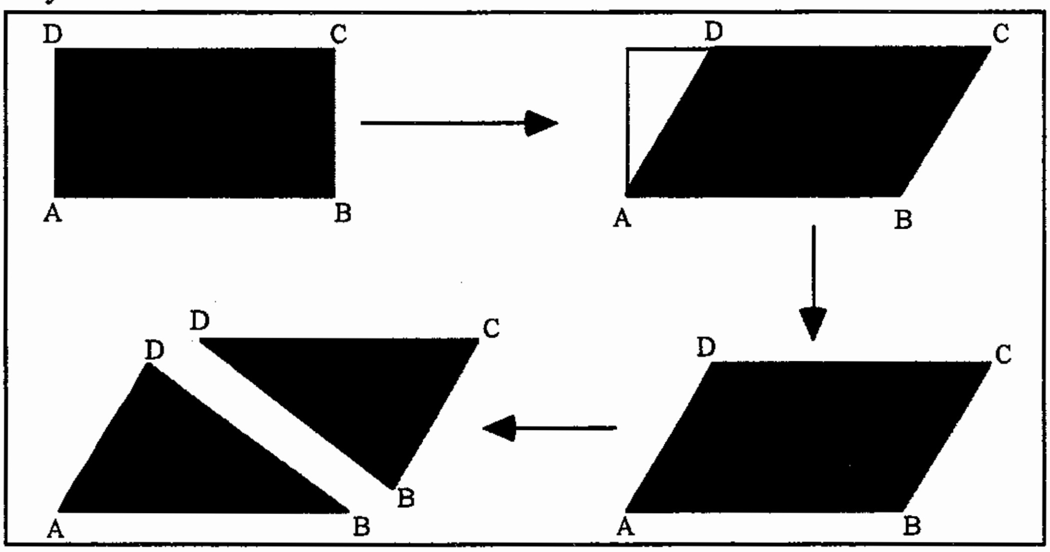
In geometry it is important that students learn to be flexible in using geometrical reasoning.

In applications students are confronted many times with two-dimensional, irregular shapes. Depending on the context of the problem, they should be able to decide how to attack it.
The following example about the sum of angles in a polygon (full text from the book) shows how students are confronted with four different ways of reasoning on this issue. In the final question (5c) there are different answers possible, depending on the type of reasoning a student chooses.

The sum of angles in a polygon

Three stories that show different ways to find the sum of angles in a polygon.

Story 1

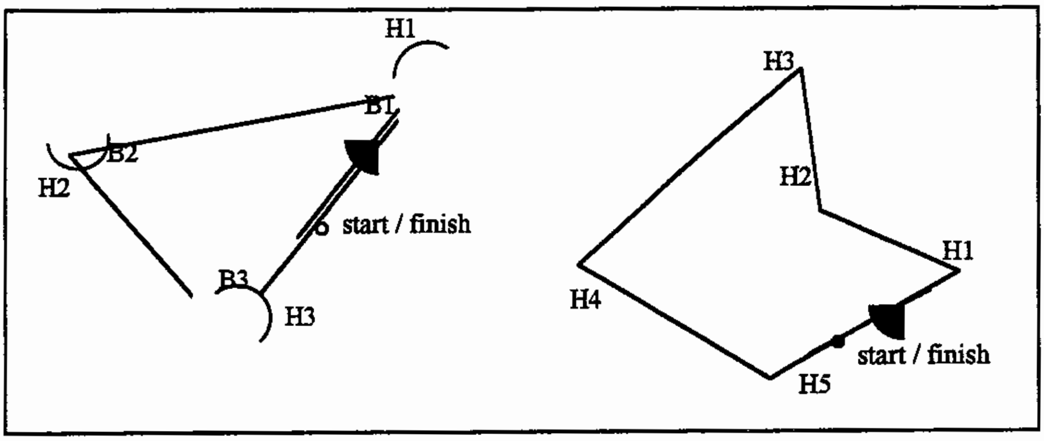


In this story there is an attempt to prove that the angles in a triangle sum up to 180_.
Starting point is the rectangle in the upper left part. The rectangle gets its name from the four right angles, so in a rectangle the four angles sum up to 360_.

1. Following the arrows in story 1, explain why you finally arrive at:
the angles in a triangle sum up to 180_

Story 2

The two pictures (page 8) both tell the story of a robot, that can be guided along the sides of any polygon. The way in which this robot is guided has two components:
* turn the robot over a given angle (to the left or to the right), while it is standing still
* move the robot along a straight line for a given distance.
We are only interested in 'turning over an angle' and not in 'walking along a line'.
First look at the picture of the triangle. The robot is standing in the point indicated with 'start'and he looks in the direction of the arrow. The robot walks to the vertex, stands still and is then turned over an angle H1. Then walk again and turn over angle H2, again walking and turn over angle H3. Finally he is back in the starting point.

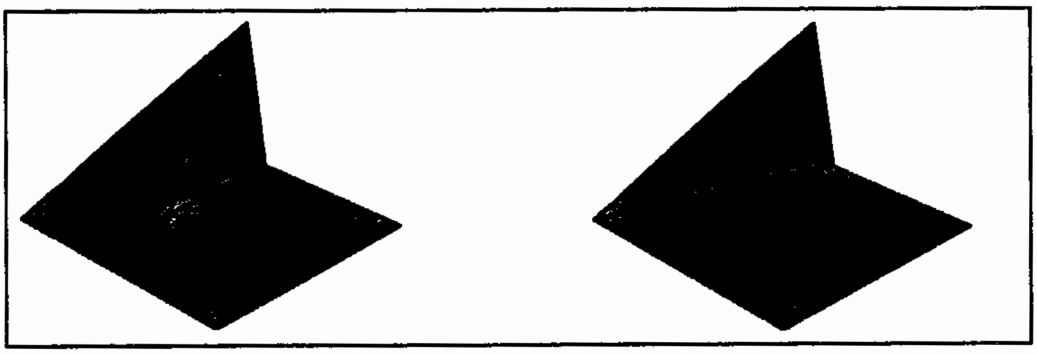


2 a. What is the sum of angles H1, H2 and H3 together?
Of course it is allowed to measure the angles to find the answer to this question, but also try to find the answer by means of reasoning.
Every turning-angle H has a neighbor-angle B, that is an angle inside the triangle.
b. How can you use the robot turning-angles H1, H2 and H3 in order to find the sum of the three angles B1, B2 and B3?

In Story 2 the figure at the right is a pentagon.
Again the robot is guided along the sides of this polygon in a similar way as it was done with the triangle.

3. Use the idea of question **2** to find the sum of the angles in the pentagon.
Be careful! There is a little problem with turning-angle H2!

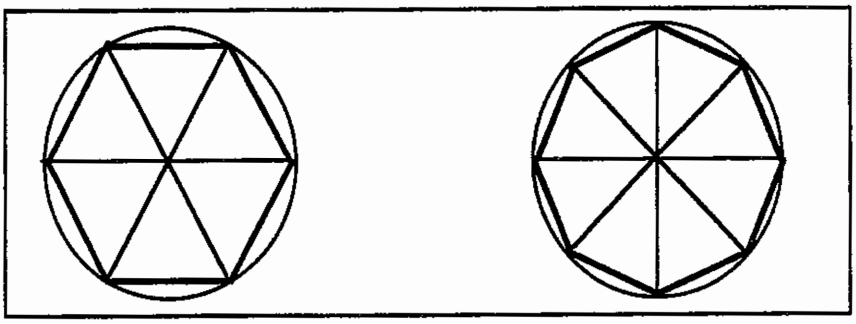
Story 3



In story 3 you see the pentagon of story 2 again. But now the pentagon is divided into triangles in two different ways.

4. In the pentagon at the left, a point inside the pentagon is chosen and connected to all five vertices.
a. Use this division to show that the five angles of the pentagon sum up to 540_
In the right pentagon one vertex is connected to the remaining non-adjacent vertices.
b. Use this division in triangles to show that the angles of the pentagon sum up to 540_

Using a circle, it is quite easy to construct regular polygons. In the following figure this is done for a regular hexagon and for a regular octagon.



5 **a.** How many degrees does one angle in a regular hexagon measure?
And in a regular octagon?
b. How about one angle in a polygon with 10 vertices? And with 12 vertices?
By continuing this story for polygons with more vertices, you can design a formula for the angle in a regular polygon with n vertices
c. Find this formula.

The answers to question **5c** vary, depending on the kind of reasoning that students choose.
Using the idea of the robot, you arrive at the formula $A = \frac{n \cdot 180^\circ - 360^\circ}{n}$
Using the idea of story 3, you arrive at $A = 180^\circ - \frac{360^\circ}{n}$
Use the regularities in a table, with entries known from earlier questions: $A = \frac{(n-2) \cdot 180^\circ}{n}$
The three formulas are the same (easily seen by mathematical professionals) but for the students they are different. Again a nice opportunity to focus on algebra, while doing geometry.

In the TWIN-project the starting point for a mathematics program is the necessity to design a curriculum that has to be supportive for vocational training. The given examples are representative for the program that is still under design.
One other important choice was the integration of the Graphing Calculator (GC) in the whole curriculum. In general there are three good reasons to introduce this tool into the curriculum.
* It helps poor algebraists to stay alive. Many of these students are not good in algebra. The GC has many features that can be used to replace algebra.
* It supports the understanding of a mathematical concept, because many things can easily be visualized. It is also possible to replace traditional ways of learning a concept by one that is supported by the GC
* It helps when students are investigating a problem, because you can easily try a number of possibilities to check a hypothesis.

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