Supporting Fraction Understanding

Marj Horne
Australian Catholic University
marj.horne@acu.edu.au

Fractions is one topic that concerns many learners. This research based workshop explores some activities that focus on the big picture understandings associated with understanding fractions. Some of the tasks have been used to provide useful assessment information for informing teaching, enabling the targeting of specific difficulties and misconceptions. The use of low cost manipulatives, games and discussion provides opportunities for learners with different intelligences to interact with the key concepts. The focus is primarily on relational understanding with procedural thinking having only a minor role. Much of this material has been used with parents and both preservice and practising teachers, who were interested in assisting children to understand and develop confidence with fractions.

Introduction
A critical question in supporting fraction understanding is to consider what a fraction is and how the learners may see fractions. Some ways of seeing fractions are:

- Part of a whole
- Part of a collection
- A division (quotient)
- A ratio (multiplicative thinking)
- A number on a number line

The development of the concept of fraction takes place over time and full understanding of the concept includes all or these ways of seeing fractions. Many children and adults have a limited understanding and only see fractions as part of whole. This limits them as it often means they do not conceive of fractions larger than one, cannot relate fractions to a number line as fractions are only part of a whole and do not recognise the link between the division sign and the form of a fraction.

Part-whole understanding
Part-whole understanding is critical in the concept development. Key ideas are that:

- Equal parts are not necessarily congruent
- Subdivision must be exhaustive
- A given fraction of a may be different to the same fraction of b

One aim is to enable the learner to move flexibly from whole to part and from part to whole. Consider the diagram in figure 1. The question is What might be the fraction? Of course there are many possible answers but each time the whole and the particular piece of interest need to be defined.
For many learners visual representations and manipulatives are important aids to learning. In the creation of models we need to ensure we are not adding to complexity while at the same time we need to provide a breadth of experiences. Models are often based on length, area of rectangles or circles. The concept of area is more difficult than length and that needs to be considered. Figure 2 shows a response to being asked to show a third. The circle is correct if the marked fraction is a fraction of the width but not if it is an area model.

The circle model can cause problems and raises some interesting questions. Do the learners see the area, the angle or the proportion of the circumference? Of course all give equivalent answers. The real problem lies in the effort to place the question in a pseudo-real context which usually involves food. For example pre-service teacher education students were asked to write some questions for a test and the lecturer then used some of the students’ questions on the test. One of the questions was as follows:

Alan, Ben, Cathy, Dana and Erin had some pizzas. Alan had 2/3 of a pizza, Ben 3/4 of a pizza, Cathy 4/5 of a pizza, Dana 6/8 of a pizza and Erin 1/2 of a pizza. Who had the most?

One student was asked this question in interview. She claimed that her friends wrote it and she had seen it before. She then drew a circle and after some time divided it into four and shaded 3/4 as shown in the left-hand circle in figure 3. The second stage was to draw another circle and again divide it into four. After some time passed she then added the lines dividing it into eight and shaded 6/8 as in the circle on the right in figure 3. She commented, with some surprise, that 3/4 and 6/8 were the same. She quite quickly drew another circle, divided it into two and shaded one half. Her last drawing was of another circle which she divided again into four. After looking at it for nearly five minutes she added the line in the lower right hand quadrant in the centre circle and shaded it as shown. She then answered that 3/4, 6/8 and 4/5 were all the same so Ben Cathy and Dana had all had the same amount of pizza.

This signals the need to think carefully about what contexts are appropriate to develop the correct concepts. While it is clear there is some lack of understanding of fractions the question that arises is how much the context may be misleading. For example everyone knows that the first to take a piece of pizza can easily obtain more pizza as the size of pieces is never uniform. The second difficulty is that circular pizzas are usually divided into eight pieces. It is not possible to have 1/5

Figure 1. Open part-whole question.

Figure 2. One third–but of what?
or 2/3 of a pizza just by taking pieces that are already cut. At least if such poor contexts for fractions are used there needs to be discussion about the problems. It is also more difficult to divide a circle into three or five than to divide a rectangle.

**Part of a collection**

For some learners the concept of fraction is restricted to a fraction of one object. Figure 4 shows one students attempt to show one third of some circles. While of course it is correct it shows that the student does not yet understand a fraction of a collection and this may be due to limited experiences.

The flexibility to move from part to whole and whole to part applies here as well. Figure 5 shows three quarters. What is two thirds?

**Some Activities to Aid Understanding**

*Fraction Strips*

This activity has assisted both adults and children to work with fractions and provides a visualisation based on length which is conceptually simpler than area based models. I have used
it particularly with parents who are interested in learning to assist their children. Teacher trainees who are unsure of the concept have also found it useful.

I usually start with strips of stiff paper or very light card that I have cut to 60 cm in length and about 2-3 cm width. Working in pairs (or sometimes individually) one strip was labelled 1 whole. Another strip, of a different colour, was folded then cut in half. The next strip was cut into three 20 cm strips. Using this approach with a combination of folds and measuring the fractions made included 1/2, 1/3, 1/4, 1/5, 1/6, 1/8, 1/10, 1/12, 1/15, and sometimes others such as 1/20 and 1/24 were added. 1/9 and 1/18 could also be used but have less common denominator connections. Participants labelled their fractions as fractions of the whole. A wall was then built with the smaller fractions lower down. I have made a similar wall on a poster with strips the same length that the class uses. The first 14 rows are shown here in figure 6. My poster has 36 rows and it includes all the fractions while the strips the learners make include only some denominators.

![Fraction Wall Diagram](image)

*Figure 6. A fraction wall.*

The poster provides an interesting pattern and students often comment on the curves.

The strips are then used in a variety of ways. It is important that the learners do not always see the long strip as the whole but recognise that there are other relationships as well. Some of the tasks include:

- Find all the different pairs where one is half of the other—how do you know that they are half? How do you know you have found them all? The discussion here should focus on two equal length parts. The word half is often misused as is well known. Children have been heard to say “It’s not fair, he/she has the bigger half” and cutting an apple “in half” really means in two roughly equal pieces.

- Find three strips that together make one whole. Write down the corresponding addition fact. Can you do it in any other way? Repeat with four strips and five strips. This is a start to the ideas of addition and the writing down of the facts found is an important step. A follow up question without the strips would be to write down two fractions (they may be different) that when added together make one. This would be followed by write a number of fraction sums that make one.

- Find three strips that together make one half. Write down the corresponding sum and repeat with different numbers of strips as in previous task.
• How many different ways can you find to cover each piece exactly with pieces of the same size? This will establish the idea of equivalence. Sometimes the learners like to write what they have discovered on one of the strips so to one of the 1/2 strips they add $= 2/4 = 3/6 = 4/8 = 5/10 = 6/12$ etc.

• Cover the whole with pieces of two different colours exactly and write the sum. Investigate in how many different ways this is possible. Repeat for three different colours. The writing of the sums each time is an important component.

• For addition place the two fractions end to end and find pieces of the one colour-denominator that together make exactly the same length. Write down the equivalence. This is shown in figure 7 for a half plus a third.

![Figure 7. Addition of fractions.](image)

Allowing the learners to do their additions this way and encouraging them to predict answers, particularly after having written the equivalent fractions on some of the strips helps them more easily move to the algorithm that is commonly used. When they have difficulty with a problem the strips and discussion with fellow learners provide the first assistance.

It is important in the addition of fractions with the strips to have some which go beyond the whole so that fractions larger than one become part of the learning as well.

_A game using the strips_

The first version is a simple one to establish the rules of the game. Using a die labelled 1/2, 1/4, 1/4, 1/8, 1/8 players take turns to roll for fractions. When two fraction strips can be traded for a single strip the trade must be made. On any turn a player may refuse to take a strip for any reason. This may be because it would be too big or might make it difficult to win. The winner is the first person to have the whole.

The more challenging version uses a die labelled 1/4, 1/3, 1/6, 1/8, 1/12, 1/12. This time it is often necessary to refuse a strip. For example if 1/6 is needed to complete a whole but 1/8 is rolled it should be rejected but it usually takes players some time to come to this realisation. Adding another rule that allows a rolled fraction to be either added, taken away or refused can enable the game to conclude more easily.

_Another fraction wall game—Colour in fractions_

This game uses a grid as shown in figure 3 and two dice. One die is labelled */2, */3, */4, */6, */8, */12 and the other 1, 1, 2, 2, 3, 3. Each player has a grid as shown in figure 8 and a pen. Players take turns to roll the two dice and then may colour in the equivalent fraction on the wall. For example, if the roll is 3 and */2 the player may colour in any combination that is equivalent to 3/2. The complete fraction rolled must be coloured or the turn missed. For example if a player has only 1/3 left to colour and rolls 3/2 the turn would be missed. The object is to be the first to colour in your own wall exactly.
Part of the discussion after the first game should be about strategies and why particular strategies are useful. A quick sketch of strips also now often assists students in answering questions.

**Fractions as division and division of fractions**

The shape of the division sign reflects the connection between fractions and division. The division sign is like a fraction with the two dots representing the numerator and denominator and the line between them, the vinculum, also between the two dots. So \( \frac{3}{5} \) is \( 3 \div 5 \). Questions such as is 10 cookies shared between 5 people can be seen as \( 10 \div 5 \). The next step is to see a question such as 3 large cookies shared between 5 people as \( 3 \div 5 \) or in fraction form as \( \frac{3}{5} \).

Questions involving division come in two main forms. Partition, which is also called sharing is where a collection is divided into equal parts. For example if a collection is sorted into three equal groups. In this form the questions can be interpreted as sharing so \( 24 \div 3 \) becomes 24 shared between three. The other form is quotation, which is also sometimes called measurement where groups or quotas are formed. For example to make groups of three. In this form the questions can be interpreted as how many so \( 24 \div 3 \) becomes 24, how many threes. This latter form of division is particularly useful when dividing fractions. For example a common error for 10 divided by \( \frac{1}{2} \) is 5, but if the question is read as 10, how many halves, the answer of 20 is easier to understand. Discussion of language and how different language presents different images is an important part of the classroom. Figure 9 shows a division of fractions problem using the strips as a visual aid.

**Figure 9.** \( \frac{1}{2} \div \frac{1}{3} \). How many of the thirds would be needed to make the half?

**Arrays**

Fraction multiplication is another fraction operation. Martin, a teenager in grade 8 who had been an excellent student in mathematics with very high grades once asked me for some assistance
with a mathematics problem. The problem was $\frac{3}{4} \times \frac{1}{2}$. He said that he had an answer and it was probably correct but he hadn’t done any fraction problems since the previous year and wanted to be sure. His answer to the problem was 3 and he had no concept of the idea that the answer should at least be less than one. I could not see how he obtained 3 so I asked him for his solution.

Well, for all fraction problems you put the fractions over a common denominator. The teacher says it is easier if you don’t worry about the lowest common denominator. You get the common denominator by multiplying the two numbers on the bottom.

$$\frac{3}{4} \times \frac{1}{2} = \frac{3 \times 1}{8}$$

Then you sort of cross multiply to get the two numbers on the top. Finally just finish the problem – 24 on top divided by 8 gives 3.

$$\frac{6 \times 4}{8} = \frac{24}{8} = 3$$

Martin has learned a procedure but without a clear understanding of its use. It works very well for addition and subtraction. Procedural knowledge (instrumental understanding) without relational knowledge (relational understanding) is limited. Students need to understand the operations and be able to relate them to other knowledge.

When Martin’s little sister in grade 1 was asked to read the question $2 \times 3$ and said “two groups of three” Martin thought for a moment and then said “Oh like $3/4$ of a group of $1/2$? That’s $3/8$.” An array or a rectangle as shown in figure 10 is another good way to illustrate this type of multiplication.

![Fraction Multiplication Illustrated](image)

**Figure 10. Fraction multiplication illustrated.**

**Benchmarking, Ordering and Fraction Sense**

Martin’s lack of appreciation for the magnitude of the answer raises the issue of building fraction sense which includes estimation of magnitude and benchmarking. Figure 11 shows a benchmarking question.
Open questions are a good way to build number sense and provide opportunities for discussion on issues. For example write down at least seven fractions that are between ½ and ¾.

Asking students to order groups of fractions, particularly if they are on cards rather than a text exercise, can reveal where there are misconceptions. For example some think the largest denominator gives the smallest fraction, which is true if the numerators are the same. Others think that it is the gap between the numerator and denominator that makes a difference. Having groups sort fraction cards and discuss their group answer is one way to assist in the overcoming the misconceptions.

**Number lines**

Too often the only interpretation people have of fractions is as part of a whole. The complete concept includes fractions as numbers which can be placed alongside whole numbers and decimals on a number line. The question below can show whether this is a problem for the learners.

Here is a number line. I am going to ask you to show me where some numbers are on the line.

Point to a quarter. I am going to ask you to show me where some numbers are on the line.

Point to two and a quarter.

There are many rich activities around that can support learners’ sense of fractions and their operations. The reference list shows a book and an article, both written for middle-school students but with ideas that I find useful at many levels.

**References**
